Algebraic Geometry of Many Particles

Denis Sunko, Department of Physics, Faculty of Science

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In the first part of the seminar, a direct connection between the classical theory of invariants and the quantum mechanics of N identical particles is explained. For the case of N fermions, such as electrons, the relevant symmetry is the Pauli principle. Wave functions of N fermions in d spatial dimensions, antisymmetric under the exchange of coordinate d-plets, are a free module over a specific ring of symmetric polynomials in $d \cdot N$ variables, less general than all symmetric polynomials, which we call the ring of bosonic excitations. With this coefficient ring the number of module generators, called shapes, is equal to $N!^{d-1}$. The shapes correspond to vacuum wave functions in physics. Analogous results hold for N conserved bosons, such as helium atoms, whose shapes must be distinguished from bosonic excitations, even if they are also symmetric functions.

The second part is devoted to the still-open problem of efficient generation of shapes for given N and d, specifically for fermions and odd d. This problem can be framed by concrete conjectures, analogous to the so-called $(N+1)^{N-1}$ and $N!$ conjectures from the 1990s, which are important solved problems of modern combinatorics [1]. Our problem appears easier than those because of the above-mentioned specialization of the coefficient ring. The unifying feature is the *diagonal action* of the symmetric group S_N on d sets of N variables. The key to the solution is also the same: to prove that certain vector spaces, which represent homogeneous submodules isomorphically, have exactly the required dimensions. This program has succeeded at least three times [1–3]. (The serial success of the same type of conjecture, requiring considerable effort anew each time, hints at some true reason, but no one seems to have stated it.) The case of even d has been studied much less than that of odd d, from which it has intriguing qualitative differences.

In the third part, a construction of a new basis of S_N-harmonic polynomials is described [4], as an intermediate step in the solution of the problem above. Its pretty symmetries enable a refinement of the coefficients of the q-factorial, the so-called Mahonian numbers, giving rise to a new integer sequence [5].

[1] M. Haiman, *Combinatorics, symmetric functions, and Hilbert schemes*, Current Developments in Mathematics, 2003 (2003) 39–111.
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[3] B. Rhoades and A. T. Wilson, *Set superpartitions and superspace duality modules*, Forum of Mathematics, Sigma 10 (2022), e105.
[4] D. K. Sunko, *Evaluation and spanning sets of confluent Vandermonde forms*, J. Math. Phys. 63, 082101 (2022).
[5] D. K. Sunko, https://oeis.org/A357611.