$n_0 = 1$ (1)

$T(1) = d > 0$

$T(n) = a \cdot T(n/b) + f(n)$, $a > 0$, $b \geq 2$, $b \in \mathbb{N}$

znamo još i $f : X \to \mathbb{R}_0^+$, tj. $f(x) \geq 0$ za $\forall x \in X$.
Dakle, u rekurziji je sve nenegativno, a početni uvjet d i faktor a su pozitivni.

$x \in X \iff x = n_0 \cdot b^i \to$ stavim $n_0 = 1$ i $x = n = b^i$, $i \in \mathbb{N}_0$

$\Rightarrow T(n) > 0$ i $T(n)$ strogo raste

– Prvo svedim ponašanje po f.
Ako je $f_1(n) \leq f_2(n)$, onda pripadni nizovi $T_1(n), T_2(n)$ zadovoljavaju $T_1(n) \leq T_2(n)$ za $\forall n \in X$.
Odavde odmah ide i (zbog nenegativnosti f, T)

$f_1 \in O(f_2)$ ili $f_1(n) \in O(f_2(n))$

$\Rightarrow T_1 \in O(T_2)$ ili $T_1(n) \in O(T_2(n))$.

– Dakle, ako je $f_1(n) \in O(\underbrace{n^p/(\log n)^{1+\varepsilon}}_{f(n)})$

onda je za pripadne T- nizove isto $T_1(n) \in O(T(n))$.

– Zato gledam "najveći mogući" f, a to je

$$\underline{f(n) = n^p/(\log n)^{1+\varepsilon}}, \quad \varepsilon > 0, \ p = \log_b a$$

i trebam pokazati da pripadni $T(n)$ zadovoljava

$$T(n) \in \Theta(n^p).$$

(Multiplikativna konst. u f nije bitna.
Isto tako, $n_0 = 1$ ili nešto drugo – nije bitno)

$$T(1) = d$$

$$T(n) = a \cdot T(n/b) + f(n) \quad , \quad n = b^i \; , \; i \geq 1$$

$$f(n) = \frac{n^p}{(\log n)^{1+q}} = \frac{n^{\log_b a}}{(\log n)^{1+q}}$$

Sad stavim $n = b^i$, $t_i = T(b^i)$.

$$f(n) = f(b^i) = \frac{b^{i \cdot \log_b a}}{(\log b^i)^{1+q}} = \frac{(\overbrace{b^{\log_b a}}^{a})^i}{(i \cdot \log b)^{1+q}} = \frac{a^i}{i^{1+q} \cdot (\log b)^{1+q}}$$

"Raspakiram" rekurziju za t_i

$$t_0 = d$$

$$t_i = a \cdot t_{i-1} + a^i \cdot \frac{1}{i^{1+q}} \cdot \frac{1}{(\log b)^{1+q}}$$

$$= a \cdot \left(a \cdot t_{i-2} + a^{i-1} \cdot \frac{1}{(i-1)^{1+q}} \cdot \frac{1}{(\log b)^{1+q}} \right) + a^i \cdot \frac{1}{i^{1+q}} \cdot \frac{1}{(\log b)^{1+q}}$$

$$= a^2 \cdot t_{i-2} + a^i \cdot \frac{1}{(\log b)^{1+q}} \cdot \frac{1}{(i-1)^{1+q}} + a^i \cdot \frac{1}{(\log b)^{1+q}} \cdot \frac{1}{i^{1+q}}$$

$$= \ldots = (\text{indukcija}) =$$

$$= a^i \cdot t_0 + \frac{a^i}{(\log b)^{1+q}} \cdot \left[\frac{1}{1^{1+q}} + \frac{1}{2^{1+q}} + \ldots + \frac{1}{(i-1)^{1+q}} + \frac{1}{i^{1+q}} \right]$$

$$= a^i \cdot \left[\underset{\substack{\uparrow \\ \text{const} \\ > 0}}{d} + \underbrace{\frac{1}{(\log b)^{1+q}}}_{\substack{\uparrow \\ \text{const}}} \cdot \left(\frac{1}{1^{1+q}} + \frac{1}{2^{1+q}} + \ldots + \frac{1}{(i-1)^{1+q}} + \frac{1}{i^{1+q}} \right) \right]$$

Sad iskoristim da je "kvaziharmonijski" red s eksponentima $1+q > 1$ oblika

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+q}}$$

<u>konvergentan</u> red, pa je $1 + \frac{1}{2^{1+q}} + \ldots + \frac{1}{i^{1+q}} < C = \text{suma tog reda}$

Dakle:

$$t_i \leq a^i \cdot \underbrace{\left[d + \frac{1}{(\log b)^{1+g}} \cdot C \right]}_{\text{const - oznaka } c}$$

$\Rightarrow$ za svaki $i \in \mathbb{N}$ je

$$t_i \leq c \cdot a^i \quad , \text{ za neku konst. } c \text{ (ne ovisi o } i)$$

Na kraju, $a^i = (b^{\log_b a})^i = b^{i \cdot \log_b a} = (b^i)^p = n^p$

pa je $(t_i = T(b^i) = T(n))$

$$T(n) \leq c \cdot n^p \quad , \text{ uvjetno za } n = b^i, \ i \in \mathbb{N}$$

ili

$$\underline{T(n) \in O(n^p)} \quad , \quad \text{— || —}$$

$T(1) = d$

$T(n) = aT(n/b) + f(n)\ ,\quad n = b^i\ ,\ i \geq 1$

$f(n) = n^{\log_b a}/\log n$

$n = b^i \qquad t_i = T(b^i)$

$$f(b^i) = b^{i\cdot\log_b a}/\log b^i = \frac{(\overbrace{b^{\log_b a}}^{a})^i}{i\cdot\log b} = \frac{1}{\log b}\cdot\frac{a^i}{i}$$

$t_0 = d$

$$t_i = a\cdot t_{i-1} + \frac{1}{\log b}\cdot\frac{a^i}{i}$$

$$= a\cdot\left(at_{i-2} + \frac{1}{\log b}\cdot\frac{a^{i-1}}{i-1}\right) + \frac{a^i}{i}\cdot\frac{1}{\log b}$$

$$= a^2 t_{i-2} + \frac{a^i}{\log b}\left(\frac{1}{i-1} + \frac{1}{i}\right)$$

$$= \ldots =$$

$$= a^i\cdot t_0 + \frac{a^i}{\log b}\underbrace{\left(1 + \frac{1}{2} + \ldots + \frac{1}{i}\right)}_{H_i\ -\ \text{harm. brojevi}}$$

$$t_i = a^i\left[d + \frac{H_i}{\log b}\right]\ ,\quad i \in \mathbb{N}_0$$

za H_i vrijedi $\qquad H_i = \Theta(\log i)$

$$\Rightarrow \quad t_i = \Theta\left(a^i\left(d + \frac{\log i}{\log b}\right)\right) \qquad \frac{c_1 + c_2\log i}{c_2\log i}\ \text{je dom. čl.}$$

ili

$$t_i = \Theta(a^i \log i)$$

vratimo: $\quad a^i = b^{i\log_b a} = n^{\log_b a} = n^p$

$$i = \frac{\log b^i}{\log b} = \frac{\log n}{\log b}$$

$$\Rightarrow \quad \boxed{T(n) = \Theta(\ n^p \log\log n\ )}$$

$$T(1) = d$$

$$T(n) = aT(n/b) + f(n), \quad n = b^i, \quad i \geq 1$$

$$f(n) = n^{\log_b a} (\log n)^{q-1}$$

$n = b^i$ $\quad t_i = T(b^i)$ $\quad f(b^i) = b^{i \log_b a} \cdot (\log b^i)^{q-1} = \underbrace{(b^{\log_b a})}_{a}{}^{i} \cdot (i \cdot \log b)^{q-1}$

$$= a^i \cdot i^{q-1} \cdot (\log b)^{q-1}$$

$$t_0 = d$$

$$t_i = a t_{i-1} + i^{q-1} \cdot a^i (\log b)^{q-1}$$

$$= a \cdot \left(a t_{i-2} + (i-1)^{q-1} (\log b)^{q-1} \cdot a^{i-1}\right) + i^{q-1} (\log b)^{q-1} \cdot a^i$$

$$= a^2 t_{i-2} + (i-1)^{q-1} (\log b)^{q-1} \cdot a^i + i^{q-1} (\log b)^{q-1} \cdot a^i$$

$$= a^2 t_{i-2} + a^i \cdot (\log b)^{q-1} \cdot \left[(i-1)^{q-1} + i^{q-1}\right]$$

$$= \dots =$$

$$= a^i t_0 + a^i (\log b)^{q-1} \cdot \left[1^{q-1} + 2^{q-1} + \dots + (i-1)^{q-1} + i^{q-1}\right]$$

$$= a^i \left[\underset{\substack{\uparrow \\ \text{const}}}{d} + (\log b)^{q-1} \cdot \underset{\uparrow}{H_i^{q-1}}\right]$$

$\to \infty$ za $i \to \infty$
pa je ovo dom. član.

$\Rightarrow$ $t_i = \Theta(a^i H_i^{q-1})$ $\quad$ iz integrala (po dijel. konst)
$H_i^{q-1} = \Theta(i^q)$

$\Rightarrow$ $t_i = \Theta(a^i i^q)$

$$= \left(a^i = b^{i \log_b a} = n^{\log_b a} = n^p\right.$$

$$i = \frac{\log b^i}{\log b} = \frac{\log n}{\log b} \Rightarrow i^q = \Theta\left((\log n)^q\right)\Big)$$

$\Rightarrow$ $t_i = \Theta\left(n (\log n)^q\right) =$

$$= \Theta\left(f(n) \log n\right)$$