[7] E. Wasserstrom, "Numerical solutions by the continuation method," *SIAM Rev.*, vol. 15, pp. 89–119, 1973.
[8] —, "Solving boundary-value problems by imbedding," *J. Ass. Comput. Mach.*, vol. 18, pp. 594–602, 1971.
[9] F. A. Ficken, "The continuation method for functional equations," *Commun. Pure Appl. Math.*, vol. 4, pp. 435–456, 1951.
[10] H. B. Keller, *Numerical Methods for Two-Point Boundary-Value Problems.* Waltham, Mass.: Blaisdell, 1968, ch. 2.
[11] E. Wasserstrom, "Identification of parameters by the continuation method," *AIAA J.*, Aug. 1973.

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Numerical Technique for the Convolution of Piecewise Polynomial Functions

ROBERT J. POLGE AND R. DUANE HAYS

***Abstract*—This paper describes a general systematic procedure for the convolution of functions that are piecewise polynomial. The procedure can be implemented by hand using a simple table format or programmed for execution on a digital computer. An algebraic convolution law is defined to replace integration; this provides an efficient digital computation algorithm. Thus, convolution operations of any complexity can be transformed into the algebraic manipulation of numbers by a digital computer.**

Examples in statistics are included. The technique is useful for the study of linear systems.

***Index Terms*—Convolution, delta function integrals, piecewise polynomial functions.**

I. Introduction

QUITE OFTEN it is difficult or impossible to obtain a mathematical expression for the convolution integral, and numerical techniques are required. Samples of the convolution integral are computed from the samples of the two functions to be convolved. Discrete convolution corresponds to the approximation of the integral by a Riemann sum. Cyclic convolution makes uses of the fast Fourier transformation to substitute a product for convolution. The computed samples cannot be exact unless both functions are bandlimited and finite.

If a continuous function can be represented by a finite number of polynomials, one for each interval, it is called piecewise polynomial. Each interval of a piecewise polynomial function is described exactly by the coefficients of the corresponding polynomial, and the entire function is completely defined by the matrix of coefficients. Piecewise polynomial functions can be used to represent, or approximate: finite duration waveforms, filter impulse responses, probability density functions and distributions, power spectra, and certain other functions of interest in radar, communications, and statistics.

This paper shows that the convolution of two piecewise polynomial functions is a piecewise polynomial function, and it presents a numerical technique to compute the set of intervals and the set of coefficients for each interval. The main advantages of the proposed technique are that: 1) it is fast,

Manuscript received August 22, 1972; revised April 24, 1973. This work was supported in part by the Advanced Ballistic Defense Missile Agency, U. S. Army, Huntsville, Ala.
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exact, and requires little storage; and 2) it is both algebraic and numerical and yields a simple mathematical expression for the resulting continuous function.

Section II shows that a piecewise polynomial function can be defined either as a set of polynomials, or by a cumulative expression valid for all times. These representations will be called, respectively, the per interval representation and the delta function integral representation. Formulas and transformation matrices are derived to provide a systematic means of going from one representation to the other.

The main purpose of Section III is to review the convolution properties of delta function integrals [1] and to develop a numerical technique for the convolution of the sum of delta function integrals.

Section IV illustrates the procedure. The technique can be implemented by hand using a simple table format, or programmed for execution on a digital computer.

In Section V, two examples are presented to show the practical use of the method for certain statistical applications. Other applications can be found in communications and radar, probability, and networks.

Section VI contains the summary and conclusions.

II. Representation of Piecewise Polynomial Functions

A function $f(t)$ is called piecewise polynomial if its domain can be divided into a finite number of intervals, over each of which $f(t)$ is a polynomial. Denote by $f(t;i)$ the expression for $f(t)$ in the ith interval, then

$$f(t) = f(t;i), \qquad \text{for } t_i \leqslant t \leqslant t_{i+1},\ 1 \leqslant i \leqslant I$$

$$f(t;i) = \sum_{m=1}^{M_i} a_{im}\, t^{m-1} \tag{1}$$

where $M_i - 1$ is the degree of $f(t;i)$ and I is the number of intervals. The representation of $f(t)$ for all time requires I polynomials such as (1), and the set $[t_i]$ which defines the lower bounds of the intervals. If M is the largest value of M_i, the per interval representation is

$$f(t;i) = \sum_{m=1}^{M} a_{im}\, t^{m-1} \tag{2}$$

where $a_{im} = 0$ for $m > M_i$. In matrix notation, (2) can be written as

$$[f(t;i)] = [a_{im}]\ [t^{m-1}] \tag{3}$$

where $i = 1, 2, \cdots, I$ and $m = 1, 2, \cdots, M$.

Another way to describe $f(t)$ in the ith interval, called cumulative representation, is to use the relation for $f(t)$ in the $(i-1)$th interval and add up a corrective polynomial starting at the ith interval:

$$f(t;i) = f(t;i-1) + \sum_{m=1}^{M} c_{im}\, \delta^{-m}(t - t_i) \tag{4}$$

where $f(t;0) = 0$, $m \geqslant 1$, and $\delta^{-m}(t)$ is the mth order integral of $\delta(t)$, the Dirac delta function. In matrix form

$$[f(t;i) - f(t;i-1)] = [c_{im}]\ [\delta^{-m}(t - t_i)] \tag{5}$$

where

$$\delta^{-m}(t - t_i) = \begin{cases} 0, & \text{for } t < t_i \\ \dfrac{(t - t_i)^{m-1}}{(m-1)!}, & \text{for } t \geqslant t_i. \end{cases} \tag{6}$$

The cumulative expression for $f(t)$ in the ith interval follows by iteration

$$f(t;i) = \sum_{l=1}^{i} \sum_{m=1}^{M} c_{lm}\, \delta^{-m}(t - t_l) \tag{7}$$

where $t_i \leqslant t < t_{i+1}$. The upper limit of l can be changed to I, because $\delta^{-m}(t - t_l) = 0$ when $t < t_l$, which occurs when $l \geqslant i + 1$. Thus

$$f(t;i) = \sum_{l=1}^{I} \sum_{m=1}^{M} c_{lm}\, \delta^{-m}(t - t_l). \tag{8}$$

This shows that (8) is valid for any interval and that the parameter i can be dropped. Therefore, using δ-function integrals, $f(t)$ can be represented by one single expression valid for all times

$$f(t) = \sum_{l=1}^{I} \sum_{m=1}^{M} c_{lm}\, \delta^{-m}(t - t_l). \tag{9}$$

The per interval representation is useful because the polynomial coefficients for each interval are computed independently; the delta function representation is useful because the piecewise function is represented by a single expression valid for all times. Transformation formulas and transformation matrices to go from one representation to the other are derived next.

Substituting (2) into (4) and differentiating $(l-1)$ times yields

$$\sum_{m=l}^{M} (a_{im} - a_{(i-1)m})\, ((m-1) \cdots (m-l+1))\, t^{m-l} = \sum_{m=1}^{M} c_{im}\, \delta^{-(m-l+1)}(t - t_i). \tag{10}$$

Let $t = t_i$, note that $\delta^{-1}(o) = 1$ and $\delta^{-k}(o) = 0$ for $k \neq 1$, and interchange l and m, then

$$c_{im} = \sum_{l=m}^{M} \frac{(l-1)!\, t_i^{l-m}}{(l-m)!} (a_{il} - a_{(i-1)l}). \tag{11}$$

Equation (11) defines a set of M linear equations that can be solved to yield

$$a_{im} - a_{(i-1)m} = \sum_{l=m}^{M} \frac{(-t_i)^{l-m}}{(l-m)!\,(m-1)!} c_{il} \tag{12}$$

where $a_{om} = 0$.

For a given interval i, (11) and (12) are easily expressed in matrix form

$$[c_{im}] = [a_{il} - a_{(i-1)l}]\ [e^1_{lm}(i)] \tag{13}$$

$$[a_{im} - a_{(i-1)m}] = [c_{il}]\ [e^2_{lm}(i)] \tag{14}$$

where $a_{om} = 0$, $e^1_{lm}(i) = e^2_{lm}(i) = 0$ for $m = 1, 2, \cdots, M$, and $l < m$, and

$$e^1_{lm}(i) = \frac{(l-1)!\, t_i^{l-m}}{(l-m)!}\ ;\ e^2_{lm}(i) = \frac{(-t_i)^{l-m}}{(l-m)!\,(m-1)!}$$

with l and m denoting rows and columns, respectively.

The following example illustrates the per interval and delta representation of a piecewise polynomial function.

Example: Let $f(t)$ be defined in the per interval notation as

$$\begin{aligned} f(t;0) &= 0, && \text{for } t < 0\\ f(t;1) &= 1 + t, && \text{for } 0 \leqslant t < 1\\ f(t;2) &= 5.4 - 4.6t + 1.2t^2, && \text{for } 1 \leqslant t < 2\\ f(t;3) &= -6 + 4.5t - 0.5t^2, && \text{for } 2 \leqslant t < 5\\ f(t;4) &= 0, && \text{for } t \geqslant 5. \end{aligned} \tag{15}$$

These relations define three matrices:

$$[f(t;i)]_{4\times 1};\qquad [a_{im}] = \begin{bmatrix} 1 & 1 & 0\\ 5.4 & -4.6 & 1.2\\ -6 & 4.5 & -0.5\\ 0 & 0 & 0 \end{bmatrix}_{4\times 3};$$

$$[t_i] = \begin{bmatrix} 0\\ 1\\ 2\\ 5 \end{bmatrix}_{4\times 1}$$

Using (11) or (13), the delta function coefficients are obtained:

$$[c_{im}] = \begin{bmatrix} 1 & 1 & 0\\ 0 & -3.2 & 2.4\\ 0 & 2.3 & -3.4\\ -4 & 0.5 & 1 \end{bmatrix}_{4\times 3}.$$

The delta function representation of $f(t)$ follows, using (9), $[t_i]$, and $[c_{im}]$,

$$\begin{aligned} f(t) &= \sum_{l=1}^{I=4} \sum_{m=1}^{M=3} c_{lm}\, \delta^{-m}(t - t_l)\\ &= \delta^{-1}(t) + \delta^{-2}(t) - 3.2\delta^{-2}(t-1) + 2.4\delta^{-3}(t-1)\\ &\quad + 2.3\delta^{-2}(t-2) - 3.4\delta^{-3}(t-2) - 4\delta^{-1}(t-5)\\ &\quad + 0.5\delta^{-2}(t-5) + \delta^{-3}(t-5). \end{aligned} \tag{16}$$

Formula (16) defines $f(t)$ at all times. Consider an interval $t_l < t < t_{l+1}$, where delta function integrals may originate at time t_l, but not within the interval. The expression for $f(t)$ in this interval is obtained by replacing, in (16), $\delta^{-m}(t - t_l)$ by zero for $t_l < t$ and by $(t - t_l)^{m-1}/(m-1)!$ for $t_l \geqslant t$. Thus, continuing the example,

$$f(t) = 0, \qquad \text{for } t \leqslant 0$$

$$f(t) = 1 + t, \qquad \text{for } 0 \leqslant t < 1$$

$$f(t) = 1 + t - 3.2(t-1) + 2.4\frac{(t-1)^2}{2}, \qquad \text{for } 1 \leqslant t < 2$$

$$\begin{aligned} f(t) &= 1 + t - 3.2(t-1) + 2.4\frac{(t-1)^2}{2}\\ &\quad + 2.3(t-2) - 3.4\frac{(t-2)^2}{2}, \qquad \text{for } 2 \leqslant t < 5 \end{aligned}$$

$$\begin{aligned} f(t) &= 1 + t - 3.2(t-1) + 2.4\frac{(t-1)^2}{2}\\ &\quad + 2.3(t-2) - 3.4\frac{(t-2)^2}{2}\\ &\quad - 4 + 0.5(t-5) + \frac{(t-5)^2}{2}, \qquad \text{for } 5 \leqslant t. \end{aligned} \tag{17}$$

These expressions can easily be simplified, and written in the format of (15). One could also go directly from the delta function representation to the per interval representation by using either (12) or (14), where

$$e^2_{lm}(i) = \begin{bmatrix} 1 & 0 & 0\\ -t_i & 1 & 0\\ t_i^2/2 & -t_i & 0.5 \end{bmatrix}.$$

Fig. 1 shows the graphical construction of $f(t)$ in the per interval and delta function integral notation. The delta function representation is cumulative with respect to time as shown, i.e., it contains terms that begin at specified times and end at infinity. The piecewise polynomial $f(t)$ is obtained by a simple algebraic addition of the component terms where $a_k\delta^{-m}(t - t_i) = a_k((t - t_i)^{m-1}/(m-1)!)$, for $t \geqslant t_i$ and zero otherwise. The advantages of the delta function integral representation are that one formula defines $f(t)$ for all t and that the convolution of two delta function integrals is a simple operation.

III. Convolution of Two Delta Function Integrals

At this point it is convenient to introduce a linear translation operator $\mathcal{T}_\tau$ defined by $\mathcal{T}_\tau f(t) = f(t - \tau)$. With this notation, the convolution of two delta function integrals can be written as

$$A_i\delta^{-m}(t - t_i) * B_k\delta^{-n}(t - t_k) = A_iB_k(\mathcal{T}_{t_i}\delta^{-m}(t) * \mathcal{T}_{t_k}\delta^{-n}(t))$$

where $*$ denotes the convolution operation. Since translation can be factored out of convolution [2], the convolution expression can be rewritten as

$$A_iB_k\,\mathcal{T}_{t_i+t_k}(\delta^{-m}(t) * \delta^{-n}(t)).$$

An equivalent expression is [2]

$$A_iB_k\,\mathcal{T}_{t_i+t_k}(\delta^{-m-n}(t) * \delta(t)).$$

Finally, noting that $\delta^{-m-n}(t) * \delta(t) = \delta^{-m-n}(t)$, the formula for convolution of two delta function integrals is obtained:

$$A_i\mathcal{T}_{t_i}\delta^{-m}(t) * B_k\mathcal{T}_{t_k}\delta^{-n}(t) = A_iB_k\,\mathcal{T}_{t_i+t_k}\delta^{-m-n}(t). \tag{18}$$

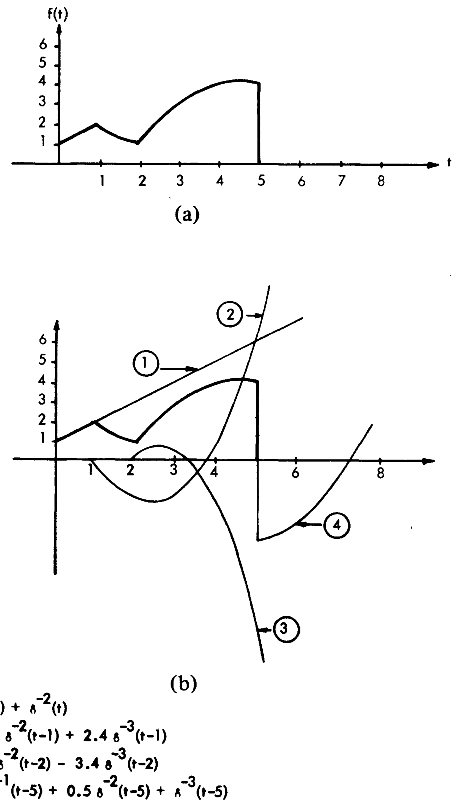

Fig. 1. Graphical construction of $f(t)$. (a) Per interval construction of $f(t)$. (b) Delta function integral construction of $f(t)$.

Each delta function integral term can be represented by a triplet consisting of its coefficient, its time tr**a**nslation, and its exponent. For example,

$$A_i \mathcal{T}_{t_i} \delta^{-m}(t) \leftrightarrow \begin{bmatrix} A_i \\ t_i \\ -m \end{bmatrix}.$$

The convolution of two delta function integrals can be written in triplet notation. Equation (18) becomes

$$\begin{bmatrix} A_i \\ t_i \\ -m \end{bmatrix} * \begin{bmatrix} B_k \\ t_k \\ -n \end{bmatrix} = \begin{bmatrix} A_iB_k \\ t_i + t_k \\ -m-n \end{bmatrix} \tag{19}$$

where the ordered triplets contain the coefficient, translation, and exponent of the delta function integral terms, respectively. Equation (19) shows that the convolution integral has been replaced by three elementary operations on the triplet elements: 1) multiplication of coefficients; 2) algebraic addition of translations; and 3) algebraic addition of exponents.

IV. Convolution of Piecewise Polynomial Functions

Formula (19) provides a very efficient technique for the convolution of piecewise polynomial functions. The technique can be implemented by hand using a simple table format, as illustrated below, or programmed for execution on a digital computer. The latter approach has been used by the authors to perform convolutions of very complicated functions.

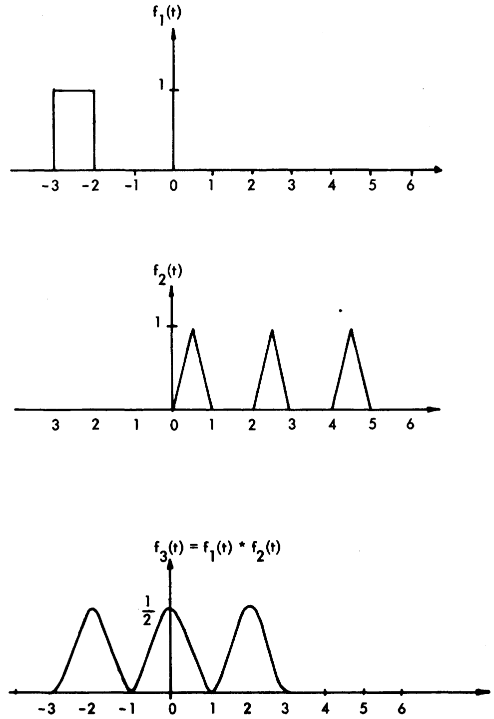

Fig. 2. Piecewise polynomial functions $f_1(t), f_2(t)$, and $f_3(t)$.

The convolution technique is best illustrated by an example. Consider the piecewise polynomial functions shown in Fig. 2. We wish to form $f_3(t)$ by convolving $f_1(t)$ with $f_2(t)$,

$$f_3(t) = f_1(t) * f_2(t).$$

The delta function representations are, respectively,

$$f_1(t) = \mathcal{T}_{-3}\delta^{-1}(t) - \mathcal{T}_{-2}\delta^{-1}(t)$$

$$\begin{aligned} f_2(t) = {} & 2\delta^{-2}(t) - 4\mathcal{T}_{0.5}\delta^{-2}(t) + 2\mathcal{T}_1\delta^{-2}(t) + 2\mathcal{T}_2\delta^{-2}(t) \\ & - 4\mathcal{T}_{2.5}\delta^{-2}(t) + 2\mathcal{T}_3\delta^{-2}(t) + 2\mathcal{T}_4\delta^{-2}(t) \\ & - 4\mathcal{T}_{4.5}\delta^{-2}(t) + 2\mathcal{T}_5\delta^{-2}(t). \end{aligned}$$

To perform the desired convolution, place the triplet representation for $f_1(t)$ along the left margin of a table, as depicted in Fig. 3, and the triplet representation of $f_2(t)$ along the top margin. Then, at the intersection of each row and column, perform the elementary operations defined by (19). Finally, triplets with equal time translations and exponents are added algebraically, and $f_3(t)$ can be written in the delta function integral representation. The sum is ordered for increasing delays and delta exponents, as shown in Fig. 3. One can easily convert $f_3(t)$ to the per interval representation, if desired, by using either (12) or (14).

This example illustrates that the convolution of two piecewise polynomial functions $f_1(t)$ and $f_2(t)$ can be obtained very simply, without integration. The procedure begins by expressing $f_1(t)$ and $f_2(t)$ in delta function integral representation. If $f_1(t)$ and $f_2(t)$ consist of M and N terms, respectively, their convolution is obtained quite easily as a sum of MN delta function integrals (or triplets) using (19). The terms are arranged in increasing times of occurrence $(t_i + t_k)$ and degree $(m + n)$; those with equal occurrence time and de-

$f_1(t)$ \ $f_2(t)$	$\begin{bmatrix}2\\0\\-2\end{bmatrix}$	$\begin{bmatrix}-4\\0.5\\-2\end{bmatrix}$	$\begin{bmatrix}2\\1\\-2\end{bmatrix}$	$\begin{bmatrix}2\\2\\-2\end{bmatrix}$	$\begin{bmatrix}-4\\2.5\\-2\end{bmatrix}$	$\begin{bmatrix}2\\3\\-2\end{bmatrix}$	$\begin{bmatrix}2\\4\\-2\end{bmatrix}$	$\begin{bmatrix}-4\\4.5\\-2\end{bmatrix}$	$\begin{bmatrix}2\\5\\-2\end{bmatrix}$
$\begin{bmatrix}1\\-3\\-1\end{bmatrix}$	$\begin{matrix}2\\-3\\-3\end{matrix}$	$\begin{matrix}-4\\-2.5\\-3\end{matrix}$	$\begin{matrix}2\\-2\\-3\end{matrix}$	$\begin{matrix}2\\-1\\-3\end{matrix}$	$\begin{matrix}-4\\-0.5\\-3\end{matrix}$	$\begin{matrix}2\\0\\-3\end{matrix}$	$\begin{matrix}2\\1\\-3\end{matrix}$	$\begin{matrix}-4\\1.5\\-3\end{matrix}$	$\begin{matrix}2\\2\\-3\end{matrix}$
$\begin{bmatrix}-1\\-2\\-1\end{bmatrix}$	$\begin{matrix}-2\\-2\\-3\end{matrix}$	$\begin{matrix}4\\-1.5\\-3\end{matrix}$	$\begin{matrix}-2\\-1\\-3\end{matrix}$	$\begin{matrix}-2\\0\\-3\end{matrix}$	$\begin{matrix}4\\0.5\\-3\end{matrix}$	$\begin{matrix}-2\\1\\-3\end{matrix}$	$\begin{matrix}-2\\2\\-3\end{matrix}$	$\begin{matrix}4\\2.5\\-3\end{matrix}$	$\begin{matrix}-2\\3\\-3\end{matrix}$

$$f_3(t) = f_1(t) * f_2(t)$$

$$= 2\mathcal{T}_{-3}\,\delta^{-3}(t) - 4\mathcal{T}_{-2.5}\,\delta^{-3}(t) + 4\mathcal{T}_{-1.5}\,\delta^{-3}(t) - 4\mathcal{T}_{-0.5}\,\delta^{-3}(t)$$

$$+ 4\mathcal{T}_{0.5}\,\delta^{-3}(t) - 4\mathcal{T}_{1.5}\,\delta^{-3}(t) + 4\mathcal{T}_{2.5}\,\delta^{-3}(t) - 2\mathcal{T}_{3}\,\delta^{-3}(t),$$

where

$$2\mathcal{T}_{-3}\,\delta^{-3}(t) \leftrightarrow \begin{bmatrix}2\\-3\\-3\end{bmatrix}, \quad -4\mathcal{T}_{-2.5}\,\delta^{-3}(t) \leftrightarrow \begin{bmatrix}-4\\-2.5\\-3\end{bmatrix}, \quad \text{etc.}$$

Fig. 3. Convolution of piecewise polynomial functions.

gree are combined algebraically. The delta function integral representation obtained for the result $f_3(t)$ is quite practical as shown by (16) and (17). However, one may prefer to have the final result in the more familiar per interval representation. This is done easily using the transformation matrix presented in Section II.

V. Applications

The formulas and techniques presented in this paper have several useful applications. This section presents two examples in statistics.

A. Probability Density for a Sum of Independent, Polynomial Distributed Random Processes

It is well known that the probability density function for a random process $r(t)$ that results from the sum of several independent random processes $x_i(t)$, $i = 1, 2, \cdots, N$, is found by convolution. Let $p_r(r)$ be the probability density function of $r(t)$ and $p_i(x_i)$ be the probability density for $x_i(t)$, $i = 1, 2, \cdots, N$. Then, if

$$r(t) = x_1(t) + x_2(t) + \cdots + x_N(t)$$

and

$$p(x_1, x_2, \cdots, x_N) = p_1(x_1)\,p_2(x_2) \cdots p_N(x_N)$$

we have

$$p_r(r) = p_1(x_1) * p_2(x_2) * \cdots * p_N(x_N).$$

Clearly, if the $p(x_i)$ are piecewise polynomial, or very nearly so, then $p_r(r)$ can be obtained very easily by repeated use of the techniques in Section IV.

For example, the result of convolving two uniform probability densities, each defined by

$$p_x(x) = \delta^{-1}(x) - \mathcal{T}_1\,\delta^{-1}(x)$$

is given by

$$p_y(x) = p_x(x) * p_x(x)$$
$$= \delta^{-2}(x) - 2\mathcal{T}_1\,\delta^{-2}(x) + \mathcal{T}_2\,\delta^{-2}(x).$$

Convolution of $p_y(x)$ with a third uniform probability density of the same form would yield

$$p_z(x) = p_x(x) * p_x(x) * p_x(x) = p_y(x) * p_x(x)$$
$$= \delta^{-3}(x) - 3\mathcal{T}_1\,\delta^{-3}(x) + 3\mathcal{T}_2\,\delta^{-3}(x) - \mathcal{T}_3\,\delta^{-3}(x)$$

and so on. Each resulting convolution is determined almost by inspection when using the tabular form of Fig. 3.

B. Power Spectrum at the Output of a Nonlinear Device

The autocorrelation $\varphi_{yy}(\tau)$ at the output of a zero-memory nonlinear device can be expressed as a polynomial in terms of the input autocorrelation function $\varphi_{xx}(\tau)$. It follows that the output power spectrum can be obtained by a combination of convolution and superposition operations on the input power spectrum.

Consider, for example, a full-wave square-law detector where the input is a narrow-band Gaussian noise,

$$x(t) = E(t)\cos(\omega_c t + \varphi(t))$$

and the output is

$$y(t) = \frac{a}{2}E^2(t) + \frac{a}{2}E^2(t)\cos 2(\omega_c t + \varphi(t))$$

or

$$y(t) = \frac{a}{2}E^2(t)$$

when the high-frequency term is neglected. The output autocorrelation functions are given by [3]

$$\varphi_{yy}(\tau) = a^2\,(2\varphi_{xx}^2(\tau) + \varphi_{xx}^2(o)).$$

The output power spectrum is obtained by Fourier transformation. Let Φ_{yy} and Φ_{xx} be the Fourier transform of φ_{yy} and and φ_{xx}, then

$$\Phi_{yy}(f) = a^2\,[2\Phi_{xx}(f) * \Phi_{xx}(f) + a^2\varphi_{xx}^2(o)\,\delta(f)].$$

In general, $\Phi_{xx}(f)$ is (or can be approximated by) a piecewise polynomial function, and $\Phi_{xx}(f) * \Phi_{xx}(f)$ is obtained easily. For example, assume that the input power spectrum $\Phi_{xx}(f)$ is made of two triangles of width B and height A, centered at frequency $-f_o$ and f_o. The condensed delta function integral representation of $\Phi_{xx}(f)$ is

$$\Phi_{xx}(f) = \frac{2A}{B}\,(\mathcal{T}_{-f_o-B} - 2\mathcal{T}_{-f_o} + \mathcal{T}_{-f_o+B} + \mathcal{T}_{f_o-B} - 2\mathcal{T}_{f_o} + \mathcal{T}_{f_o+B})\,\delta^{-2}(f)$$

where $\delta^{-2}(f)$ has been symbolically factored out because it belongs to every term. It follows that

$$\Phi_{xx}(f) * \Phi_{xx}(f) = \frac{4A^2}{B^2}\left(\mathcal{I}_{-2f_o-B} - 2\,\mathcal{I}_{-2f_o} + \mathcal{I}_{-2f_o+B} + 2\mathcal{I}_{-B} - 4\mathcal{I}_o + 2\mathcal{I}_B + \mathcal{I}_{2f_o-B} - 2\mathcal{I}_{2f_o} + \mathcal{I}_{2f_o+B}\right)\delta^{-4}(f)$$

from which $\Phi_{yy}(f)$ can be written in δ-function form or in per interval notation.

VI. Summary and Conclusions

This paper presents an algebraic technique for the convolution of functions that are piecewise polynomial. The functions are expressed as a sum of delta function integrals, where each delta function integral is represented as a triplet. Convolution reduces to simple operations on triplets. Two examples showing its practical application to problems in statistics are presented.

Transformation matrices are derived to go from the per interval representation of piecewise polynomial functions to the cumulative delta function representation, and vice versa.

While a piecewise polynomial is a continuous function, it is represented by a finite set of triplets. The convolution of two piecewise polynomial functions can be pictured in a table that shows the combinations of all the triplets. The result of the convolution is a piecewise polynomial function that is obtained by multiplying or adding triplet components, instead of performing a complicated integration. Such operations can be performed numerically on a digital computer.

Digital computer programs have been written to accommodate functions of high complexity, and to perform successive convolutions as required in some applications. In particular, if f_1 is a function represented by M polynomial pieces and f_2 is represented by N polynomial pieces, then $f_3 = f_1 * f_2$ requires MN polynomial pieces, in general. Direct computation of f_3 would require the evaluation of MN integrals, and becomes rapidly impractical. On the other hand, the procedure outlined in this paper allows the convolution to be performed systematically, and efficiently, on a digital computer. Moreover, the computations are exact, and in many applications, are more efficient than cyclic convolution using the fast Fourier transform.

References

[1] R. Bracewell, *The Fourier Transform and Its Applications.* New York: McGraw-Hill, 1965.

[2] R. J. Polge and L. Callas, "Simplification of sequences of operators," *IEEE Trans. Syst., Man, Cybern.*, vol. SMC-2, pp. 526–533, Sept. 1972.

[3] W. W. Harman, *Principles of the Statistical Theory of Communication.* New York: McGraw-Hill, 1963.

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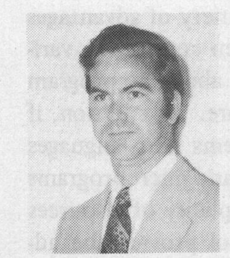

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