Corollary 8: For $S_{\beta_i}^{n_i}$, $1 \le i \le k$, let γ_i be the greatest common root of δ and β_i when they are commensurable, and let γ_i be unity otherwise. Let

$$m \le \min_i \{(n_i - 1) \log_\delta \beta_i + \log_\delta \gamma_i\} \tag{25}$$

and let $Q_i = R_{\beta_i}^{n_i} R_\delta^m$ for $1 \le i \le k$. If $Q = Q_j Q'$ for any $1 \le j \le k$ where Q' is composed from the mappings Q_i, $1 \le i \le k$, then $Q = Q_j$.

The fact that m must be bounded from above in (25) in order to guarantee control of accumulated error and avoid situations such as that exhibited in Fig. 10 demonstrates that the phrase "carry more digits" does not always mean that greater overall accuracy will follow, and such cliches should not be used as a substitute for a true understanding of the formal structure of floating-point number systems and base conversion.

References

[1] D. W. Matula, "Base conversion mappings," *1967 Spring Joint Computer Conf., AFIPS Proc.*, vol. 30. Washington, D. C.: Thompson, 1967, pp. 311–318.

[2] ——, "The base conversion theorem," *Proc. Amer. Math. Soc.*, vol. 19, no. 3, pp. 716–723, June 1968.

[3] ——, "In-and-out conversions," *Commun. ACM*, vol. 11, no. 1, pp. 47–50, January 1968.

[4] ——, "Towards an abstract mathematical theory of floating-point arithmetic," *1969 Spring Joint Computer Conf., AFIPS Proc.*, vol. 34. Montvale, N. J.: AFIPS Press, 1969, pp. 765–772.

[5] N. Metropolis and R. L. Ashenhurst, "Radix conversion in an unnormalized arithmetic system," *Math Computation*, vol. 19, pp. 435–441, 1965.

[6] L. Collatz, *Functional Analysis and Numerical Mathematics* (translated by H. Oser). New York: Academic Press, 1966.

[7] M. Urabe, "Roundoff error distribution in fixed-point multiplication and a remark about the rounding rule," *SIAM J. Numer. Anal.*, vol. 5, no. 2, pp. 202–210, 1968.

[8] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 3rd ed. Oxford, England: Clarendon Press, 1954, p. 373.

[9] I. B. Goldberg, "27 bits is not enough for 8-digit accuracy," *Commun. ACM*, vol. 10, pp. 105–106, February 1967.

The Correspondence Between Methods of Digital Division and Multiplier Recoding Procedures

JAMES E. ROBERTSON, MEMBER, IEEE

***Abstract*—This paper relates previous analyses of the binary SRT division to the theory of multiplier recoding. Since each binary quotient digit has three possible values, the quotient resulting from the SRT division is in recoded form; in this paper it is shown that the recoding is a function of the divisor, and the method for determining the characteristic Boolean function of the recoding is presented. The relationship between the division and the recoding is established by scaling the division in such a way that the scaled "divisor" becomes a constant. Higher radix results are also discussed.**

***Index Terms*—Binary arithmetic, division, minimal representations, multiplication, multiplier recoding, redundancy.**

Introduction

STATISTICAL analyses of the so-called SRT[1] method of binary division have been conducted by Freiman [1][2] and Shively [8]. At each recursive step of this division procedure, three alternatives are possible; shift left, add and shift left, or subtract and shift left. One may therefore take the point of view that each quotient digit correspondingly has one of the three values 0, -1, or $+1$. Thus the division procedure results in a quotient in redundant recoded form. It is the first purpose of this paper to establish the correspondence between the quotient recodings and a class of multiplier recodings.

It is first necessary to review some aspects of the theory of multiplier recoding. For the binary case with recoded digital values of $+1$, 0, or -1, a recoding can in general be characterized by the choice of two Boolean functions. For the important class of arithmetically symmetric recodings (defined below), the two Boolean functions are duals of one another; hence this class of recodings is characterized by the choice of one Boolean function. A third class of recodings is next defined by restricting the Boolean functions in such a way that each function can be determined by the choice of a single binary numerical parameter in the interval 0 to 1. It is this third class of recodings that corresponds to the quotient recodings of the SRT division.

The next step in establishing the correspondence is that of scaling. It is obvious that the value of the quotient remains unchanged if both the divisor and dividend are multiplied

Manuscript received August 18, 1969; revised December 18, 1969. This work was supported in part by the National Science Foundation under Grant NSF-GP-4636.

The author is with the Department of Computer Science, University of Illinois, Urbana, Ill. 61801.

[1] The earliest published description of a binary division involving redundancy in the representation of the quotient is contained in [4].

[2] Numbers in brackets refer to articles listed in the Reference section.

by the same parameter; similarly the detailed rules for each recursive step can be scaled in such a way that both the value and the particular recoded form of the quotient remain unchanged [3]. One further observation is that, although in practice the divisor for the SRT division is restricted to the range from 1/2 to 1, the method is valid for all divisors greater than 1.

Although the primary emphasis here is on the binary case, with redundancy limited to the use of three digital values -1, 0, and $+1$, procedures and examples are described for establishing a correspondence with known division methods involving greater redundancy with radix 2, and with higher radix division methods.

Review of the Theory of Multiplier Recoding

A multiplier y is represented in radix r by a sign digit y_0 and m nonsign digits $y_1, y_2, \cdots, y_m$. The algebraic value of a fraction y in radix complement representation is then

$$y = -y_0 + \sum_{i=1}^{m} r^{-i} y_i.$$

The multiplier y is said to be in conventional form if the sign digit y_0 is either 0 or 1 and if each nonsign digit y_i has one of the r values $0, 1, 2, \cdots, r-1$.

The effect of recoding is to transform y into m digits y_i' $(i=1, 2, \cdots, m)$, with each y_i' selected from the extended set of values $-(r-1), \cdots, -1, 0, 1, \cdots, r-1$ or from a subset thereof. If the extended set of values over which y_i' may range has more than r values, the recoded representation is redundant.

The basic equation for multiplier recoding follows from the observation that the addition of a mode digit m_i in digital position i is compensated for by the subtraction of rm_i in digital position $i+1$. The net effect at the ith digital position is then

$$y_i' = y_i + m_i - rm_{i-1}. \tag{1}$$

The requirement that the algebraic value of y shall remain unchanged by recoding is expressed as

$$-y_0 + \sum_{i=1}^{m} r^{-i} y_i = \sum_{i=1}^{m} r^{-i} y_i' = y,$$

and by substitution of (1), leads to the boundary mode conditions $m_0 = y_0$ and $m_m = 0$. Restriction of the range of y_i' to $-(r-1), \cdots, -1, 0, 1, \cdots, r-1$ restricts each m_i to one of the two values 0, 1.

The more usual practice in multiplication is to inspect first the least significant digit y_m and direct attention thereafter in a serial fashion to digits of greater significance to the left of y_m. The recodings considered here are right-directed; that is, the most significant digits are inspected first, and attention is directed thereafter serially to digits of lesser significance to the right.

For a right-directed recoding, (1) is applied recursively with i assuming the values $1, 2, \cdots, m$, in ascending order. For each value of i, the values of m_{i-1} and y_i are known, and m_i and y_i' are to be determined.

For the binary case $(r=2)$, each y_i and m_i have one of the values 0, 1 and may be treated as Boolean variables, and y_i' has one of the three values $-1, 0, 1$. The sign of y_i' is m_{i-1}, the magnitude of y_i' is $y_i \oplus m_i$ (where $\oplus$ is the symbol for EXCLUSIVE OR); therefore

$$y_i' = (-1)^{m_{i-1}} (y_i \oplus m_i). \tag{2}$$

The unknown mode digit m_i can be determined by the Boolean equation

$$m_i = m_{i-1} \bar{y}_i \vee \bar{y}_i f_i \vee m_{i-1} g_i \tag{3}$$

in which $m_{i-1} g_i$, for example, means m_{i-1} AND g_i, the symbol $\vee$ represents the INCLUSIVE OR, and $\bar{y}_i$ means NOT y_i. The Boolean functions f_i and g_i are in theory arbitrary functions of conventional multiplier digits other than y_i. For the right-directed recodings discussed by Penhollow [5], the functional dependence of f_i and g_i was restricted to digits y_{i+j}, for $j \geq 1$, that is, on multiplier digits to the right of y_i.

It is sometimes convenient to impose the restriction of arithmetic symmetry on the recoding. A recoding is arithmetically symmetric if the recoded representation of $-y$ can be found from the recoded representation of y by replacing in each digital position 0's by 0's, $+1$'s by -1's, and -1's by $+1$'s. The effect of arithmetic symmetry on (3) is that f_i and g_i are dual Boolean functions, i.e.,

$$g_i = f_i^D.$$

The Simplest Minimal Right-Directed Binary Recoding

A binary recoding is minimal if the probability that y_i' is nonzero is minimal. The simplest such right-directed recoding [5] is obtained if, in (3),

$$f_i = y_{i+1} y_{i+2}; \qquad g_i = f_i^D = y_{i+1} \vee y_{i+2}. \tag{4}$$

Since $g_i = f_i^D$, the recoding is arithmetically symmetric.

The tabular equivalent of (2) and (3), with f_i and g_i defined by (4), is given in Table I. Table II presents the numerical example of the recoding of the binary fraction 45/256 $= 0.00101101$, using the rules embodied in Table I.

Division as a Method of Recoding

The recursion relationship applicable to many varieties of division is

$$X_i = rX_{i-1} - q_i d, \tag{5}$$

in which r is the radix, i is an index ranging from 1 to m, the number of quotient digits, d is the divisor, X_{i-1} and X_i are successive partial remainders, and q_i is the ith quotient digit. The initial partial remainder X_0 is the dividend. The basic problem in division is that of selection of q_i; this selection is based on the values of the shifted partial remainder rX_{i-1} and the divisor d.

The problem of selection of q_i is eased if q_i is redundantly represented, that is, if each q_i may have more than r values. The rules for selection of q_i, coupled with (5), characterize

TABLE I

THE SIMPLEST MINIMAL BINARY RIGHT-DIRECTED RECODING

m_{i-1}	y_i	y_{i+1}	y_{i+2}	m_i	y_i'
0	1	0/1*	0/1	0	1
0	0	1	1	1	1
0	0	1	0	0	0
0	0	0	1	0	0
0	0	0	0	0	0
1	1	1	1	1	0
1	1	1	0	1	0
1	1	0	1	1	0
1	1	0	0	0	$\bar{1}$†
1	0	0/1	0/1	1	$\bar{1}$

* 0/1 indicates that the digit may be either 0 or 1.
† $\bar{1}=-1$.

TABLE II

RECODING OF 45/256. THE RESULT OF THE RECODING IS $0.001100\bar{1}\bar{1}=45/256$.

i	Digits Inspected	Corresponding Values	Result of Recoding	Comments
1	$m_0 y_1 y_2 y_3$	0 0 0 1	$m_1=0$ $y_1'=0$	1
2	$m_1 y_2 y_3 y_4$	0 0 1 0	$m_2=0$ $y_2'=0$	
3	$m_2 y_3 y_4 y_5$	0 1 0 1	$m_3=0$ $y_3'=1$	
4	$m_3 y_4 y_5 y_6$	0 0 1 1	$m_4=1$ $y_4'=1$	
5	$m_4 y_5 y_6 y_7$	1 1 1 0	$m_5=1$ $y_5'=0$	
6	$m_5 y_6 y_7 y_8$	1 1 0 1	$m_6=1$ $y_6'=0$	
7	$m_6 y_7 y_8 y_9$	1 0 1 0	$m_7=1$ $y_7'=\bar{1}$	2
8	$m_7 y_8 y_9 y_{10}$	1 1 0 0	$m_8=0$ $y_8'=\bar{1}$	2, 3

Comments: 1) $m_0=y_0$ is the left boundary condition.
2) y_9 and y_{10} are assumed to be zero.
3) The right boundary condition $m_8=0$ is satisfied.

TABLE III

SRT DIVISION OF 15/128 BY 2/3. THE RESULT OF THE DIVISION IS $45/256=0.001100\bar{1}\bar{1}$.

$2\,x_0$	$=0.001111$		15/64
$x_1=2\,x_0$	$=0.001111$	$q_1=0$	
$2\,x_1$	$=0.01111$		15/32
$x_2=2\,x_1$	$=0.01111$	$q_2=0$	
$2\,x_2$	$=0.1111$		15/16
$-d$	$=1.0101\underline{01}$	$q_3=1$	
$x_3=2\,x_2-d$	$=0.010\underline{001}$		13/48
$2\,x_3$	$=0.10\underline{001}$		13/24
$-d$	$=1.010\underline{10}$	$q_4=1$	
$x_4=2\,x_3-d$	$=1.11100$		$-3/24=-1/8$
$2\,x_4$	$=1.1100$		$-1/4$
$x_5=2\,x_4$	$=1.1100$	$q_5=0$	
$2\,x_5$	$=1.100$		$-1/2$
$x_6=2\,x_5$	$=1.100$	$q_6=0$	
$2\,x_6$	$=1.000$		-1
$+d$	$=0.10\underline{10}$	$q_7=\bar{1}$	
$x_7=2\,x_6+d$	$=1.10\underline{10}$		$-1/3$
$2\,x_7$	$=1.0\underline{10}$		$-2/3$
$+d$	$=0.1\underline{01}$	$q_8=\bar{1}$	
$x_8=2\,x_7+d$	$=0.000$		0

a method of division, aside from special procedures preliminary to and following the recursive steps.

Division may be regarded as a recoding procedure if the resultant quotient is represented in other than conventional form. Let

$$Q' = \frac{Qd}{d}, \tag{6}$$

where Q is the number to be recoded. Both the divisor d and the product Qd are in conventional form; division by d then produces Q', an algebraically equivalent but recoded version of Q.

The simplest division with redundancy in the representation of the quotient is the binary SRT method. With $r=2$, each quotient digit q_i may have one of the three values -1, 0, or $+1$. The selection rules are particularly simple; they are:

1) if $-1/2 \le 2X_{i-1} < 1/2$, $q_i=0$;
2) if $d \ge 0$ and $1/2 \le 2X_{i-1}$, then $q_i=1$, and if $d \ge 0$ and $2X_{i-1} < -1/2$, then $q_i=-1$;
3) if $d<0$ and $2X_{i-1} < -1/2$, then $q_i=1$, and if $d<0$ and $1/2 \le 2X_{i-1}$, then $q_i=-1$.

Although the method is normally used with the divisor normalized (i.e., either $1/2 \le d < 1$ or $-1 \le d < -1/2$), the method remains valid if the upper limit on the magnitude of the divisor is removed (i.e., either $d \ge 1$ or $d < -1$).

The recoding equivalent to that of Table II is obtained if, in (6), $d=2/3$, $Q=45/256$, and $Qd=15/128$. The division Qd/d illustrated in Table III, using the rules of the SRT division, yields a quotient Q', each digit q_i of which is identical with the corresponding digit y_i' resulting from the right-directed recoding of Tables I and II. The division example of Table III is artificial in the sense that the divisor $d=2/3$ cannot be represented in a binary computer of finite precision. In the table, 2/3 is represented as $0.10\underline{10}$ or a variant thereof, with the length of the stroke indicating the period of the repetitive pattern.

THE SCALED DIVISION

Many of the properties of a division method remain invariant under scaling by an arbitrary multiplicative factor z, provided that not only (5) but also the rules for selection of quotient digits are scaled by the same factor z.

Fig. 1 illustrates the mapping of $|rX_{i-1}|$ onto $|X_i|$ for the SRT division with $d=2/3$. The analysis of the division of Fig. 1 is similar to that of Freiman, but differs in two important respects.

1) The interval of shifted partial remainder magnitudes (i.e., of $|2X_{i-1}|$) is (0, 1) rather than (1/2, 1).
2) One step corresponds to the generation of one quotient digit, rather than an addition or subtraction followed by normalization.

The quantity 1/2, the abscissa of the discontinuity in the mapping function in Fig. 1, is called the comparison constant since the comparison of $|2X_{i-1}|$ with this constant

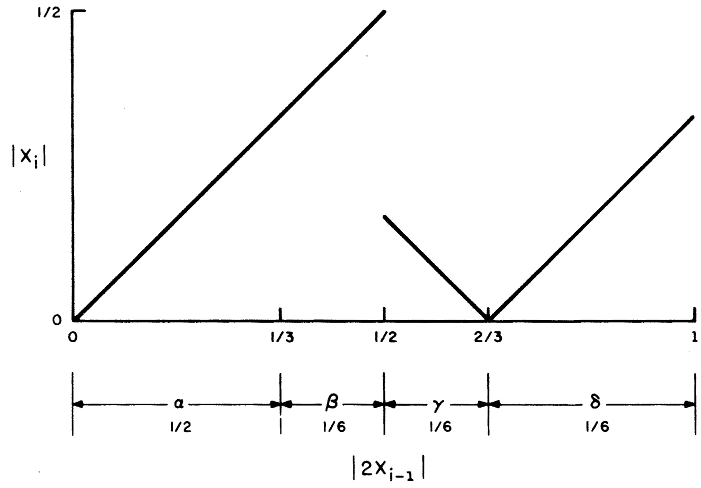

Fig. 1. Conventional SRT division with $d=2/3$.

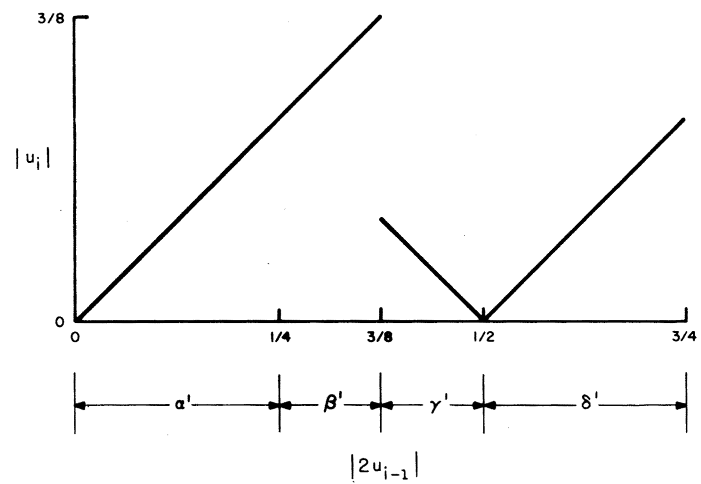

Fig. 2. Scaled SRT division.

determines whether q_i is zero or ± 1. The comparison constant plays the following important role in the analysis of the SRT division. If $(1/2)^-$ and $(1/2)^+$ are defined by

$$\left(\frac{1}{2}\right)^- = \lim_{\Delta\to 0}\left(\frac{1}{2} - \Delta\right), \quad \left(\frac{1}{2}\right)^+ = \lim_{\Delta\to 0}\left(\frac{1}{2} + \Delta\right), \quad \Delta \geq 0,$$

then endpoints of intervals are 1/2 and the images of $(1/2)^-$ and $(1/2)^+$, namely 0, 1/3, 1/2, 2/3, and 1. The intervals thus found are α, β, γ, and δ and the steady-state probability that $|2X_{i-1}|$ lies within each interval is 1/2 for α and 1/6 for β, γ, and δ. The probability that $|q_i|$ is 1 is then the probability that $|2X_{i-1}|$ is in the interval (1/2, 1) and is 1/3. Since the shift average $\langle s\rangle$ is the reciprocal of this probability, $\langle s\rangle=3$.

Fig. 2 shows the effect of scaling Fig. 1 by the factor $z=1/2d=3/4$.

If the scaled partial remainder is denoted by u_i, then $u_i=zX_i$ for each value of i. Thus each partial remainder, including the dividend x_0, is scaled. The divisor scales to 1/2, and the comparison constant scales from 1/2 in Fig. 1 to $1/2z=3/8$ in Fig. 2. The images of the comparison constant, i.e., the endpoints of the intervals, are also scaled by the factor z. The probabilities associated with the intervals remain invariant; the probability densities, due to the scaling on interval lengths, are scaled by the factor z^{-1}. Equation (6), with $z=1/2d$ scales to $Q'=(1/2)Q/1/2$, and the quotient Q' resulting from the scaled division are identical not only in algebraic value but also in digit pattern to the quotient resulting from the conventional SRT division.

In effect, the conventional SRT division is characterized by a comparison constant which is always 1/2; the divisor value is the parameter that determines the properties of the recoding. The choice of the scaling factor $z=1/2d$ results in a procedure in which the "divisor" is always 1/2 and the comparison "constant" $(1/2)z$ becomes the parameter which determines the properties of the recoding. More complicated methods of division involve addition or subtraction of more than one divisor multiple of the form nd, and perhaps values of the radix r greater than 2. (The radix r is an integer, and n is usually an integer, but is sometimes a simple rational

TABLE IV
SCALED SRT DIVISION WITH $(1/2)z=3/8$

x_0	$= 0.000101101$	
$2\ x_0$	$= 0.00101101$	
$x_1 = 2\ x_0$	$= 0.00101101$	$q_1 = 0$
$2\ x_1$	$= 0.0101101$	
$x_2 = 2\ x_1$	$= 0.0101101$	$q_2 = 0$
$2\ x_2$	$= 0.101101$	
$-d$	$= 1.1$	$q_3 = 1$
$x_3 = 2\ x_2 - d$	$= 0.001101$	
$2\ x_3$	$= 0.01101$	
$-d$	$= 1.1$	$q_4 = 1$
$x_4 = 2\ x_3 - d$	$= 1.11101$	
$2\ x_4$	$= 1.1101$	
$x_5 = 2\ x_4$	$= 1.1101$	$q_5 = 0$
$2\ x_5$	$= 1.101$	
$x_6 = 2\ x_5$	$= 1.101$	$q_6 = 0$
$2\ x_6$	$= 1.01$	
$+d$	$= 0.1$	$q_7 = \bar{1}$
$x_7 = 2\ x_6 + d$	$= 1.110$	
$2\ x_7$	$= 1.100$	
$+d$	$= 0.1$	$q_8 = \bar{1}$
$x_8 = 2\ x_7 + d$	$= 0.000$	

fraction.) For these divisions, the scaling factor is $z=1/rd$, and the "divisor multiples" are, after scaling, of the form n/r.

The scaled version of the SRT division example of Table III is presented in Table IV. The scaling factor is $z=1/2d=3/4$, the dividend is $(1/2)Q=45/512$, the "divisor" is 1/2, and the comparison constant is $(1/2)z=3/8$. The detailed rules for selection of quotient digits are those given for the SRT method, with the comparison constant 1/2 (and its negative) replaced by 3/8 (and its negative) and with each partial remainder X_{i-1} replaced by the corresponding scaled partial remainder u_{i-1}. Note that each quotient digit resulting from the scaled division is identical to the corresponding digits of Tables II and III. Equation (5) becomes $u_i=2u_{i-1}-(1/2)q_i$, with the dividend $u_0=(1/2)Q=45/512$.

THE CORRESPONDENCE BETWEEN MULTIPLIER RECODING AND THE SCALED DIVISION

The first step in establishing the correspondence is to show that the basic equation for multiplier recoding (1), when augmented by the weighted sum of multiplier digits y_{i+j}, $1 \leq j \leq m-i$, can be interpreted as the recursion relationship for the scaled division. The second step is to translate the detailed rules for selection of quotient digits into Boolean functions or an equivalent statement of the rules for a right-directed recoding.

Equation (1) may be rewritten as

$$-m_{i-1} + r^{-1}y_i = r^{-1}(y_i' - m_i) \tag{7}$$

in which, for a right-directed recoding, m_{i-1} and y_i are known, and y_i' and m_i are to be determined. Adding

$$\sum_{j=1}^{m-i} r^{-j-1}y_{i+j}$$

to both sides of (7) yields

$$-m_{i-1} + \sum_{j=0}^{m-i} r^{-j-1}y_{i+j} = r^{-1}\left(y_i' - m_i + \sum_{j=1}^{m-i} r^{-j}y_{i+j}\right). \tag{8}$$

If ru_{i-1} is defined as

$$ru_{i-1} = -m_{i-1} + \sum_{j=0}^{m-i} r^{-j-1}y_{i+j},$$

then

$$ru_i = -m_i + \sum_{j=0}^{m-(i+1)} r^{-j-1}y_{i+j+1} = -m_i + \sum_{j=1}^{m-i} r^{-j}y_{i+j}.$$

Substitution in (8) yields

$$ru_{i-1} = r^{-1}(y_i' + ru_i),$$

or

$$u_i = ru_{i-1} - r^{-1}y_i'. \tag{9}$$

Equation (9) can be interpreted as the recursion relationship for division, (5), scaled so that the divisor becomes $1/r$; that is, $z=1/rd$. The quantity ru_{i-1} is identified with the scaled shifted partial remainder, which is known, u_i is the partial remainder to be determined, and y_i' is identified as the quotient digit to be determined by the selection rules. The mode digit m_i is identified as the sign digit of the partial remainder u_i.

The above analysis indicates that (5) for division and (1) for recoding are of the same order of generality. Just as it is necessary to augment (5) with a specific set of selection rules in order to completely characterize a division, it is also necessary to augment (1) with specific rules for determining the values of recoded digits. One of the advantages of the correspondence established here is now obvious. Sets of selection rules augmenting (5) are known for a wide variety of division procedures, including use of a higher radix than the binary, and including use of quotient digit values which are not integers. On the other hand, rules augmenting (1) have been limited to radix 2 with recoded digit values of -1, 0, and 1. Thus, the translation of the quotient digit selection rules into rules for right-directed recodings, to be discussed in connection with specific methods of division in examples to follow, can be expected to provide additional insight into the theory of right-directed recodings.

THE CLASS OF RIGHT-DIRECTED RECODINGS CORRESPONDING TO THE SCALED SRT DIVISION

The analysis of the previous section indicates that for a binary radix the scaled partial remainder $2u_{i-1}$ is related to the Boolean variables employed during recoding by the equation

$$2u_{i-1} = -m_{i-1} + \sum_{j=0}^{m-i} 2^{-j-1}y_{i+j}.$$

Equation (3), for an arithmetically symmetric right-directed recoding, is

$$m_i = m_{i-1}\bar{y}_i \vee \bar{y}_i f_i \vee m_{i-1}f_i^P. \tag{10}$$

Thus, $2u_{i-1}$ contains all the information necessary for the determination of m_i since the functional dependence of f_i and f_i^P is on the $y_{i+j} (j \geq 1)$. Furthermore, the quotient digit resultant from the division is equivalent to the recoded multiplier digit y_i', which is easily determined once m_i is known, from the equation

$$y_i' = (-1)^{m_{i-1}}(y_i \oplus m_i).$$

It remains to be shown how, given the quotient digit selection rules, the function f_i can be determined.

The quotient digit selection rules for the scaled SRT division with the scaling factor $z=1/2d$ becomes

$$1) \text{ if } -\frac{z}{2} \leq 2u_{i-1} < \frac{z}{2}, y_i' = 0 \quad (11a)$$

$$2) \text{ if } \frac{z}{2} \leq 2u_{i-1}, y_i' = 1 \quad (11b)$$

$$3) \text{ if } 2u_{i-1} < -\frac{z}{2}, y_i' = -1. \quad (11c)$$

These rules, with the recursion relationship [(9) with $r=2$], namely

$$u_i = 2u_{i-1} - \tfrac{1}{2}y_i', \quad (12)$$

completely describe the scaled division. In these equations, the scaling factor z is a parameter in the range $0 \leq z \leq 1$, corresponding to $\infty \geq d \geq 1/2$. The choice of z determines f_i (and its dual) for substitution in (10). Note that (11) implies arithmetic symmetry; hence the use of f_i^D in (10) rather than the arbitrary function g_i of (3). The proof of arithmetic symmetry follows from the fact that if two divisions are performed with scaled dividends u_0 and $-u_0$, (11) guarantees that in the two resultant quotients, corresponding digits will either both be zero or negatives of one another.

Given the value of z, the function f_i can be determined in the following way. Express z as a binary fraction of the form $2^{-k}Z$, where k is an integer $(0 \leq k \leq \infty)$ and Z is an odd integer such that $0 \leq Z \leq 2^k$. f_i is then a function of k digits y_{i+j}, for $j=1, 2, \cdots, k$, and has $2^k - Z$ terms in its canonical expansion. Each minterm has k digits; the jth digit may be either y_{i+j} or $\bar{y}_{i+j}$ and is assigned a weight w_j equaling 1 and 0, respectively. The nth minterm itself can be associated with a weight W_n,

$$W_n = \sum_{j=1}^{k} 2^{-j} w_j,$$

which never exceeds $1-2^{-k}$. The canonical expansion of f_i is then the disjunction of all minterms with weights in the range $z \leq W_n \leq 1-2^{-k}$. As an example of the determination of f_i, consider the simplest minimal right-directed recoding, which corresponds to a scaled SRT division of Table IV, with $z=3/4$. For $z=3/4$, $Z=3$, $2^k=4$, and $k=2$. f_i is then a function of y_{i+1} and y_{i+2}, and has $2^k - Z = 1$ minterm. The four minterms involving y_{i+1} and y_{i+2} are, respectively, $\bar{y}_{i+1}\bar{y}_{i+2}$, $\bar{y}_{i+1}y_{i+2}$, $y_{i+1}\bar{y}_{i+2}$, and $y_{i+1}y_{i+2}$ with respective weights 0, 1/4, 1/2, and 3/4. The only acceptable minterm with weight $\geq z = 3/4$ is the last one, therefore $f_i = y_{i+1}y_{i+2}$ and $f_i^D = y_{i+1} \vee y_{i+2}$.

Analysis of the SRT Family of Right-Directed Recodings

The SRT division has been intensively analyzed by Freiman and Shively. The crucial step of identifying the Markov chain as the mathematical model of the SRT division was made by Freiman. In Shively's model, one time step corresponds to the generation of one quotient digit, including zero values with partial remainder magnitudes in the range [0, 1]. In contrast, one time step in Freiman's model corresponds to the generation of one *nonzero* quotient digit with as many partial remainder normalizations as are required, and partial remainder magnitudes are in the interval [1/2, 1]. Although Shively's results were originally given for the SRT division with the divisor magnitude $|d|$ in the range [1/2, 1], and with $|d|$ as the independent variable, his results are restated here for the *scaled* SRT division with the scaling factor $z=1/2d$ as the independent variable, with z in the range [0, 1]. Thus, the trivial extension of Shively's results to include the divisor range $1 < |d| \leq \infty$ (or $0 \leq z < 1/2$) is presented here.

Shively's results are summarized with reference to Fig. 3, which shows the location of the lower order discontinuities (see 4a, 4b below) in the steady-state probability density of the scaled partial remainder magnitude u as functions of the scaling factor z.

1) The range of u is $[0, 1-z]$ for $0 \leq z \leq 1/2$ and is $[0, z]$ for $1/2 \leq z \leq 1$. Therefore, the probability density is zero outside these ranges, namely in the triangular region bounded by $u=1$, $u=Z$, and $u=1-z$.

2) The probability density $p(u, z)$ is symmetric about $u=1/2$; for a fixed value z_0 of z,

$$p\left(\frac{1}{2} + \theta, z_0\right) + p\left(\frac{1}{2} - \theta, z_0\right) = 2 \quad \text{for } 0 < \theta < \frac{1}{2}. \quad (13)$$

Symmetry of this form implies that a plot of the locations of discontinuities in the probability density in the u, z plane for $u > 1/2$ is the mirror image of the plot of discontinuities for $u < 1/2$. Furthermore, for each value z_0 of z, if $p(1/2+\theta, z_0)$ is known, $p(1/2-\theta, z_0)$ can be determined from (13). In particular, symmetry and result 1, above, imply that the probability density of u is uniformly 2 for the triangular region, bounded by $u=0$, $u=z$, and $u=1-z$.

3) For fixed $z=z_0$, $p(u, z_0)$ is a monotonic function of u for $0 < u < 1$. This result guarantees that $p(u, z_0)$ is uniform between negative jumps in the value of $p(u, z_0)$ as u increases. The next two results are concerned, respectively, with deterministic procedures for finding the locations and magnitudes of the discontinuities in $p(u, z)$. Some indication of the complexity of the pattern of discontinuities is given by Fig. 4.

4) The locations of the discontinuities in $p(u, z)$ are found as follows.

a) The zeroth order pair of discontinuities is $u_0' = 1-z$ and $u_0 = z$. In the triangular region of Fig. 3 bounded by $u=1$, $u=1-z$, and $u=z$, $p(u, z)=0$. Since $p(u, z)$ is positive and nonzero elsewhere in the unit square, there is a negative jump in $p(u, z)$ in the upper half $(u > 1/2)$ at $u=1-z (z \leq 1/2)$

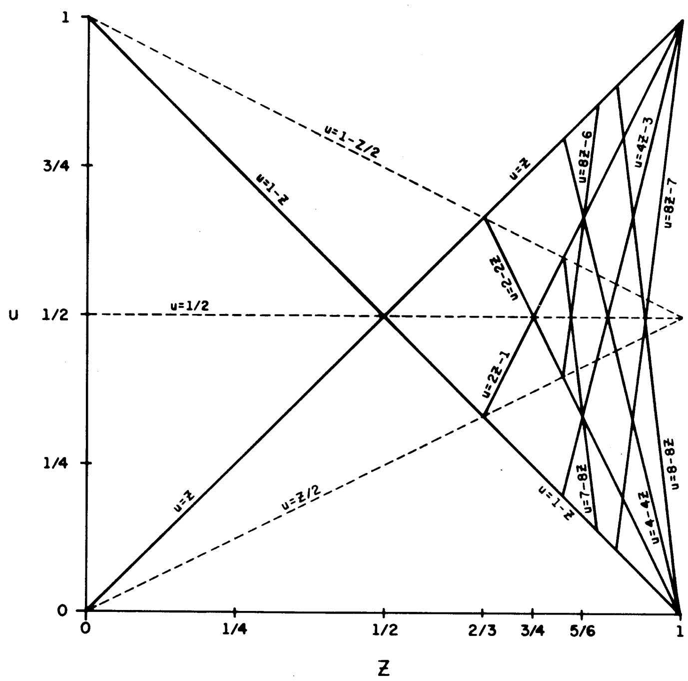

Fig. 3. Discontinuities of order three or less.

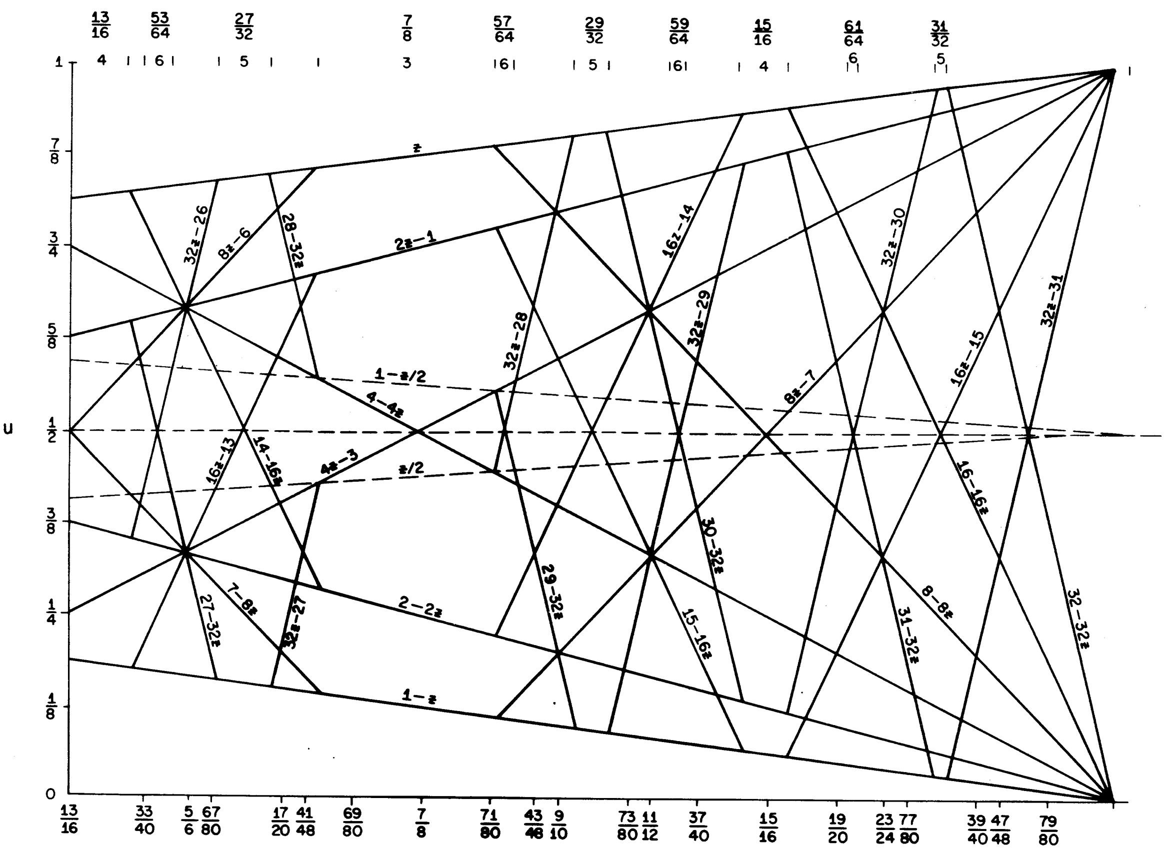

Fig. 4. Discontinuities of order five or less.

and at $u=z(z\geq 1/2)$. By symmetry, there is a negative jump in $p(u, z)$ in the lower half $(u<1/2)$ at $u=z(z\leq 1/2)$ and at $u=1-z(z\geq 1/2)$.

b) Given the kth order pair of discontinuities, with the equations $u_k=2^k z-b_k$ and $u'_k=(b_k+1)-2^k z$, the $(k+1)$st order pair is formed as follows [cf. (11), (12)].

i) If $u_k<z/2$, then $u_{k+1}=2u_k=2^{k+1}z-2b_k$. For $u'_k>1-z/2$, $u'_{k+1}=(2b_k+1)-2^{k+1}z$.

ii) If $u'_k<z/2$, then $u'_{k+1}=2u'_k=(2b_k+2)-2^{k+1}z$. For $u_k>1-z/2$, $u_{k+1}=2(u_k-1/2)=2^{k+1}z-(2b_k+1)$.

Thus, b_{k+1} is either $2b_k$ or $2b_k+1$, depending on whether u_k or u'_k is less than $z/2$. Note that the two lines forming each pair of lines of discontinuity are symmetric. The two lines are symmetric if $u_k+u'_k=1$; since the rules for generating pairs of lines result in $u_{k+1}+u'_{k+1}=1$, symmetry is preserved.

c) For an interval of z, the process of generating pairs of lines of discontinuity terminates when u_k and u'_k are in the interval $(z/2, 1-z/2)$ of u. Let the value of k for the terminating pair of lines of discontinuity be $T-1$. The lines are $u_{T-1}=2^{T-1}z-b_{T-1}$ and $u'_{T-1}=(b_{T-1}+1)-2^{T-1}z$. These lines intersect at $u=1/2$, $z=(2b_{T-1}+1)/(2^T)$, and the interval of z for which no additional lines of discontinuity can occur is

$$\frac{2b_{T-1}}{2^T-1}<z<\frac{2(b_{T-1}+1)}{2^T+1}.$$

Examples are the following.

T	u_{T-1}	u'_{T-1}	Intersection with $u=1/2$	Interval of z
1	z	$1-z$	$z=1/2$	$0<z<2/3$
2	$2z-1$	$2-2z$	$z=3/4$	$2/3<z<4/5$
4	$8z-6$	$7-8z$	$z=13/16$	$4/5<z<14/17$
3	$4z-3$	$4-4z$	$z=7/8$	$6/7<z<8/9$
4	$8z-7$	$8-8z$	$z=15/16$	$14/15<z<16/17$

5) Symmetry requires that the magnitudes of the jumps at each of the two discontinuities of a symmetric pair be equal. If the magnitude of the jump at u_k is a_k, then the sum for the symmetric pair u_k, u'_k is $2a_k$. The magnitudes of the jumps form a geometric sequence; that is, $a_{k+1}=(1/2)a_k$. For an interval of z having T pairs of discontinuities, the jump magnitudes are

$$a_k=\frac{2^{-k}}{2(1-2^{-T})}\qquad k=0,1,\cdots,T-1. \tag{14}$$

Equation (14) follows from the requirement that

$$\sum_{k=0}^{T-1} 2a_k=2,$$

since the total change in the probability density from $u=0$ to $u=1$ is 2.

6) The probability $P_0(z)$ that the recoded digit y'_i is zero is

$$P_0(z)=\int_0^{z/2} p(u, z)du$$

$$P_0(z)=z \quad \text{for } 0\leq z\leq 2/3$$

$$P_0(z)=2/3 \quad \text{for } 2/3\leq z\leq 5/6.$$

$P_0(z)$ decreases in a nonmonotonic way from 2/3 to 1/2 in the interval $5/6\leq z\leq 1$.

The probability $P_1(z)$ that the magnitude of the recoded digit $|y'_i|$ is 1 is

$$P_1(z)=1-P_0(z)$$

and the shift average $\langle s\rangle$ is

$$\langle s\rangle=\frac{1}{P_1(z)}.$$

Shively's results, as restated above, form the basis for the analysis of the family of right-directed recodings corresponding to the scaled SRT division. The recodings which have attracted the greatest attention in the past are the minimal ones, for which $P_1(z)$ has its minimum value of 1/3. The minimal recodings of the SRT family correspond to a choice of z in the range $2/3\leq z\leq 5/6$. The simplest minimal recoding of Table I is obtained by choosing z to be the simplest binary fraction in the minimal range; namely, $z=3/4$, which yields $f_i=y_{i+1}y_{i+2}$. Other relatively simple minimal recodings correspond to the choice of $z=11/16$, or $f_i=y_{i+1}(y_{i+2}\vee y_{i+3}y_{i+4})$ and to the choice of $z=13/16$ or $f_i=y_{i+1}y_{i+2}(y_{i+3}\vee y_{i+4})$. (These expressions for the f_i have been simplified.)

The minimal recodings of the SRT family are a subset of the minimal right-directed recodings found by Penhollow. The Penhollow minimal recodings differ in two ways.

1) For the SRT family, if a minterm of weight W_k is included in the canonical expansion of f_i, all minterms having weight greater than W_k are also included. This restriction does not apply to the Penhollow recodings, for example,

$$f_i=y_{i+1}(y_{i+2}y_{i+3}\vee y_{i+3}y_{i+4}\vee y_{i+2}y_{i+4})$$

yields a minimal right-directed recoding which is not a member of the SRT family, since the minterms of the canonical expansion of f_i have weights 11/16, 13/16, 7/8, and 15/16. The minterm of weight 3/4, needed in the canonical expansion for SRT minimum recoding with $k=4$ and $z=13/16$, is not included in the canonical expansion of f_i.

2) The restriction of arithmetic symmetry was not imposed by Penhollow. Let F_i be the class of all minimal right-directed recoding functions f_i, and let G_i be the class of all dual functions f_i^D. Then any function from the class F_i and any function from the class G_i can be used as f_i and g_i, respectively, in (3) to yield a minimal Penhollow recoding. Arithmetic symmetry, in contrast, requires that once f_i is chosen, the one member of class G_i which is f_i^D must be used as g_i, rather than any member of class G_i.

Other recodings of the SRT family of historic interest are the following.

1) $z=0$, $f_i=1$, $f_i^p=0$. In this case, the recoding equation reduces to $m_i=\bar{y}_i$, and each y_i' is either $+1$ or -1. The corresponding division method is nonrestoring division.

2) $z=1/2$, $f_i=f_i^p=y_{i+1}$. This recoding corresponds to differentiation in the following physical sense. Assume that a magnetic surface is magnetized positively for each digit y_i which is one, and is magnetized negatively for each digit which is zero. The recoding corresponds to the pulse pattern obtained as the magnetic surface is moved past a reading head if a positive pulse is associated with $+1$, a negative pulse with a -1, and the absence of a pulse with zero.

This recoding is also related to the transformation from binary to the Gray code.

3) $z=1$, $f_i=0$, $f_i^p=1$. The recoding equation reduces to $m_i=m_{i-1}$, so that $y_i'=y_i$ if $m_0=y_0=0$, and $y_i'=-\bar{y}_i$ if $m_0=y_0=1$. Thus positive numbers are left unchanged by this recoding, and a negative number is replaced by the digitwise negative of its diminished radix complement.

4) $z=2/3$. The recoding for $z=2/3$ is the canonical recoding discussed by Reitwiesner [6], and shown to be the simplest minimal left-directed recoding by Penhollow. The fact that 2/3 cannot be represented by a finite number of binary digits indicates that, as a right-directed recoding, it would be necessary to inspect all digits to the right of y_i.

The Right-Directed Recoding Corresponding to the STRETCH Division With $d=2/3$

In the STRETCH division [2], additional redundancy is introduced into the representation of the quotient in such a way that the probability that the quotient digit is zero, and hence the shift average, is increased. The divisor d, and the multiples $(3/4)d$ and $(3/2)d$ are available for addition or subtraction at each step; hence the possible values for each quotient digit are $-3/2$, -1, $-3/4$, 0, 3/4, 1, and 3/2.

In addition to the comparison constant $k=1/2$ to determine if another step of normalization can be performed, two more comparison constants are required. Ideally $k=7/8|d|$ would be employed to separate the partial remainder range for which $|q_i|=3/4$ from the partial remainder range for which $|q_i|=1$. Similarly $k=5/4|d|$ would be used to separate the range for which $|q_i|=1$ from the range for which $|q_i|=3/2$.

With $d=2/3$, and after scaling so that the scaled divisor is 1/2, the equation relating successive partial remainders is

$$x_i = 2x_{i-1} - \frac{1}{2}y_i'$$

and the rules for selection of y_i' are

$$\text{if } 2x_{i-1} < -\frac{5}{8}, y_i' = -\frac{3}{2}$$

$$\text{if } -\frac{5}{8} \le 2x_{i-1} < -\frac{7}{16}, y_i' = -1$$

$$\text{if } -\frac{7}{16} \le 2x_{i-1} < -\frac{3}{8}, y_i' = -\frac{3}{4}$$

TABLE V

The Recoding Corresponding to the STRETCH Division

m_{i-1}	y_i	y_{i+1}	y_{i+2}	y_{i+3}	y_i'	m_i	y_{i+1}^*	y_{i+2}^*
0	1	1			3/2	0	0	
0	1	0	1	1	3/2	1	1	1
0	1	0	1	0	3/2	1	1	1
0	1	0	0	1	1	0	0	0
0	1	0	0	0	1	0	0	0
0	0	1	1	1	1	1	1	1
0	0	1	1	0	3/4	0	0	0
0	0	1	0	1	0	0	1	0
0	0	1	0	0	0	0	1	0
0	0	0			0	0	0	
1	1	1			0	1	1	
1	1	0	1	1	0	1	0	1
1	1	0	1	0	0	1	0	1
1	1	0	0	1	−3/4	1	1	1
1	1	0	0	0	−1	0	0	0
1	0	1	1	1	−1	1	1	1
1	0	1	1	0	−1	1	1	1
1	0	1	0	1	−3/2	0	0	0
1	0	1	0	0	−3/2	0	0	0
1	0	0			−3/2	1	1	

$$\text{if } -\frac{3}{8} \le 2x_{i-1} < \frac{3}{8}, y_i' = 0$$

$$\text{if } \frac{3}{8} \le 2x_{i-1} < \frac{7}{16}, y_i' = \frac{3}{4}$$

$$\text{if } \frac{7}{16} \le 2x_{i-1} < \frac{5}{8}, y_i' = 1$$

$$\text{if } \frac{5}{8} \le 2x_{i-1}, y_i' = \frac{3}{2}.$$

The tabular equivalent of the rules of the recoding are given in Table V. Note that the introduction of fractional values of y_i' requires that the original digits y_{i+1} and y_{i+2} be replaced by the values y_{i+1}^* and y_{i+2}^* in subsequent steps of the recoding.

A particular numerical example of this recoding procedure is the following:

$$\begin{array}{l} 0\cdot 1\ 0\ 1 \quad 0\ 1\ 0\ 1\ 1\ 0\ 1\ 1 \quad 1\ 0\ 0\ 1\ 0\ 0\ 0 \\ \cdot 3/2\ 0\ 0\ -3/2\ 0\ 3/4\ 0\ 0\ 1\ 0\ 0\ -3/4\ 0\ 0\ \bar{1}. \end{array}$$

The probability of a zero after recoding is 3/4, corresponding to a shift average of 4.

A Right-Directed Decimal Recoding

As an example of a higher radix recoding, a decimal right-directed recoding can be derived from the decimal division example of [7]. In this example, two serial steps determine, first, $q_{i+1}'=\bar{5}$, 0, 5 with comparison constants ideally equal to $\pm 2.5d$, and second, $q_{i+1}''=\bar{2}$, $\bar{1}$, 0, 1, 2 with comparison constants ideally equal to $\pm 0.5d$ and $\pm 1.5d$. The quotient digit q_{i+1} is then

$$q_{i+1} = q_{i+1}' + q_{i+1}''.$$

TABLE VI
A DECIMAL RECODING EXAMPLE

$10u_0$	0.1415926535	$0.05 \leq 10u_0 < 0.15$ $q_1 = 1$
$10u_1$	0.415926535	$0.35 \leq 10u_1 < 0.45$ $q_2 = 4$
$10u_2$	0.15926535	$0.15 \leq 10u_2 < 0.25$ $q_3 = 2$
$10u_3$	-0.4073465	$-0.45 \leq 10u_3 < -0.35$ $q_4 = \bar{4}$
$10u_4$	-0.073465	$-0.15 \leq 10u_4 < -0.05$ $q_5 = \bar{1}$
$10u_5$	$+0.26535$	$0.25 \leq 10u_5 < 0.35$ $q_6 = 3$
$10u_6$	-0.3465	$-0.35 \leq 10u_6 < -0.25$ $q_7 = \bar{3}$
$10u_7$	-0.465	$10u_7 < -0.45$ $q_8 = \bar{5}$
$10u_8$	$+0.35$	$0.35 \leq 10u_8 < 0.45$ $q_9 = 4$
$10u_9$	-0.5	$10u_9 < -0.45$ $q_{10} = \bar{5}$
10_{10}	0	

After scaling by the factor $1/10d$, and combining the two serial steps into one, the rules for the scaled division, for each scaled partial remainder u_i, become

$$\begin{aligned}
&\text{if } 10u_i < -0.45, q_{i+1} = \bar{5}\\
&\text{if } -0.45 \leq 10u_i < -0.35, q_{i+1} = \bar{4}\\
&\text{if } -0.35 \leq 10u_i < -0.25, q_{i+1} = \bar{3}\\
&\text{if } -0.25 \leq 10u_i < -0.15, q_{i+1} = \bar{2}\\
&\text{if } -0.15 \leq 10u_i < -0.05, q_{i+1} = \bar{1}\\
&\text{if } -0.05 \leq 10u_i < +0.05, q_{i+1} = 0\\
&\text{if } 0.05 \leq 10u_i < 0.15, q_{i+1} = 1\\
&\text{if } 0.15 \leq 10u_i < 0.25, q_{i+1} = 2\\
&\text{if } 0.25 \leq 10u_i < 0.35, q_{i+1} = 3\\
&\text{if } 0.35 \leq 10u_i < 0.45, q_{i+1} = 4\\
&\text{if } 0.45 \leq 10u_i, \; q_{i+1} = 5.
\end{aligned}$$

The example of the recoding of 0.1415926535, treated here as a scaled division with a divisor value of 0.1, and with the recursion relationship $u_{i+1} = 10u_i - 0.1q_{i+1}$, is shown in Table VI.

The result of the recoding is $0.142\bar{4}\bar{1}3\bar{3}\bar{5}4\bar{5}$. The recoding procedure requires inspection of the sign and first decimal digit, and knowledge of whether or not the second decimal digit is greater than or equal to five.

SUMMARY

This report establishes a correspondence between results in binary recording theory obtained by Penhollow [5] and the analyses of binary division by Freiman [1] and Shively [8]. The correspondence is advantageous for the following reasons.

1) The results of the binary division analyses become applicable to recoding theory.
2) Other known methods of division, when subjected to similar correspondence relationships, yield new methods of recoding applicable to multiplication procedures.
3) For the author, the correspondences have emphasized the differences between multiplication and division. One essential difference is that a division requires a (theoretically infinite) family of recodings of the quotient, whereas in multiplication one member of the family usually suffices.

REFERENCES

[1] C. V. Freiman, "Statistical analysis of certain binary division algorithms," *Proc. IRE*, vol. 49, pp. 91–103, January 1961.

[2] O. L. MacSorley, "High speed arithmetic in binary computers," *Proc. IRE*, vol. 49, pp. 67–91, January 1961.

[3] G. Metze, "A class of binary divisions yielding minimally represented quotients," *IRE Trans. Electronic Computers*, vol. EC-11, pp. 761–764, December 1962.

[4] M. Nadler, "A high speed electronic arithmetic unit for automatic computing machines," *Acta Technica*, no. 6, pp. 464–478, 1956.

[5] J. O. Penhollow, "A study of arithmetic recoding with applications in multiplication and division," Ph.D. dissertation, University of Illinois, Urbana, Ill., September 1962. (Also, University of Illinois, Urbana, Ill., Rept. 128, DCL, September 10, 1962.)

[6] G. W. Reitwiesner, "Binary arithmetic," in *Advances in Computers*, vol. 1, F. L. Alt, Ed. New York: Academic Press, 1960.

[7] J. E. Robertson, "A new class of digital division methods," *IRE Trans. Electronic Computers*, vol. EC-7, pp. 218–222, September 1958.

[8] R. R. Shively, "Stationary distribution of partial remainders in S-R-T digital division," Ph.D. dissertation, University of Illinois, Urbana, Ill., June 1963. (Also, University of Illinois, Urbana, Ill., Rept. 136, DCL, May 15, 1963.)

[9] T. D. Tocher, "Techniques of multiplication and division for automatic binary computers," *Quart. J. Mech. Appl. Math.*, vol. 11, pt. 3, pp. 364–384, 1958.