A Method for Solving Polynomial Equations by Continued Fractions

AMNON BRACHA-BARAK, MEMBER, IEEE

Abstract—**A method for the approximation of all the real roots of an n-order polynomial equation is developed. It is assumed that intervals containing the solutions are known. Bilinear transformations are used to approximate the solution. Convergence is achieved.**

Index Terms—**Bilinear transformation, continued fractions, quadratic equation, Riccati equation, selection rules.**

I. INTRODUCTION

IN THIS paper we generalize earlier results by the same author [1] and develop them for finding all the zeros of an n-order polynomial equation. In [1] it was shown that for a limited class of functions, such as quadratic or cubic equations, a solution can be approximated by a continued fraction of the form

$$\frac{A_k}{B_k} = \frac{p_1}{q_1 +} \frac{p_2}{q_2 +} \cdots \frac{p_k}{+ q_k} \tag{1}$$

where A_k and B_k are determined from the recursion

$$\begin{aligned} A_i &= q_i A_{i-1} + p_i A_{i-2} \\ B_i &= q_i B_{i-1} + p_i B_{i-2} \qquad i = 2,3,\cdots \end{aligned} \tag{2}$$

with initial values:

$$\begin{aligned} A_0 &= 0 \quad A_1 = p_1 \\ B_0 &= 1 \quad B_1 = q_1. \end{aligned}$$

The digit set for p_i and q_i were selected as simple binary constants, e.g., $\frac{1}{2}$ or 1, in order to reduce the amount of time required to evaluate (2).

In the current paper we show that polynomial equations of any order can be transformed by a bilinear transformation such as $x_i = p_i/(q_i + x_{i+1})$ into another polynomial equation of the same order, where a simple recursion exists between the coefficients of the two polynomials.

The result is that if a method for selecting p_i and q_i for the ith step can be developed, then we can approximate the solution by using recursion (2). The selection method is described in Section V.

In Section IV we develop a method for selecting two constants, a and b for p_i, and q_i and we show the interval of solutions that can be approximated by these two constants. By using different pairs of constants we can, therefore, approximate different solutions of the given equation. Theorem 1 gives a proof of convergence, an important step in the development of the method. Rate of convergence is discussed in detail in [1].

Manuscript received July 14, 1972; revised April 23, 1973. This work was supported in part by the National Science Foundation under Grant US NSF GJ813.

The author was with the Department of Computer Science, University of Illinois at Urbana-Champaign, Urbana, Ill. 61801. He is now with the Department of Computer Science, Hebrew University of Jerusalem, Jerusalem, Israel.

II. BILINEAR TRANSFORMATIONS

Following the analysis of Wynn [4] we define a continued fraction as a sequence of bilinear transformations of the form

$$f_k = \frac{p_k}{q_k + f_{k+1}}, \qquad k = 1,2,\cdots, \tag{3}$$

where $f_k(x)$ is a function of x and p_k, q_k are constants. The resulting continued fraction is

$$\begin{aligned} f_1 &= \frac{p_1}{q_1 +} \frac{p_2}{q_2 +} \cdots \frac{p_n}{+ q_n + f_{n+1}} \\ &= \frac{A_n + f_{n+1}A_{n-1}}{B_n + f_{n+1}B_{n-1}}, \qquad n = 1,2,\cdots, \end{aligned}$$

where the functions A_i and B_i satisfy the recursion (2).

In a recent paper by the author [1] it was shown, following a note of Wynn [4], that the solution of the Riccati equation

$$y' + ay^2 + by + c = 0,$$

where a, b, c are functions of x or constants, can be expanded as a continued fraction by using a series of bilinear transformations of the form (3), and such that each function f_k, $k = 1,2,\cdots$ also satisfies the Riccati equation. In [1] the recursions for the kth Riccati equation were developed and a method for selecting p_k, q_k for each step was shown for several functions.

In the current paper we develop the results of [1] for polynomials of any order, and we show how the method can be used to find the zeros of these polynomials.

One important assumption in the development of the method in [1] was that p_k, q_k are simple binary constants, i.e., 1 or $\frac{1}{2}$, and therefore the various recursions that are involved require only "short" operations. In the current paper we generalize this result and include other values for p_k and q_k.

III. THE SOLUTION OF A POLYNOMIAL AS A CONTINUED FRACTION

We show now how distinct roots of a given polynomial can be approximated by using bilinear transformations.

First we develop the recursion for the coefficients of the polynomial for the $i+1$th step by using the coefficient of the ith step, $i = 1,2,\cdots$.

Let

$$P_n{}^1(x) = \sum_{k=0}^{n} a_k{}^1 x_1{}^k = 0 \tag{4}$$

be a given polynomial equation of order n. The index 1 refers to the first step in the recursion. In particular, the index 1 is *not* an exponent, whereas the index k is an exponent when used as a superscript.

We use a substitution of the form

$$x_i = \frac{p_i}{q_i + x_{i+1}}, \qquad i = 1,2,\cdots, \tag{5}$$

where p_i,q_i are constants to be defined.

Suppose that the ith step polynomial equation is of the form

$$P_n{}^i = a_n{}^i x_i{}^n + a_{n-1}{}^i x_i{}^{n-1} + \cdots + a_1{}^i x_i + a_0{}^i = 0, \tag{6}$$

then by using substitution (5) we have

$$a_n{}^i\left(\frac{p_i}{q_i + x_{i+1}}\right)^n + a_{n-1}{}^i\left(\frac{p_i}{q_i + x_{i+1}}\right)^{n-1} + \cdots + a_1{}^i \frac{p_i}{q_i + x_{i+1}} + a_0{}^i = 0.$$

Multiply by $(q_i + x_{i+1})^n$ to normalize the coefficient of $a_n{}^i$. We get

$$a_n{}^i p_i{}^n + a_{n-1}{}^i p_i{}^{n-1}(q_i + x_{i+1}) + \cdots + a_1{}^i p_i (q_i + x_{i+1})^{n-1} + a_0{}^i (q_i + x_{i+1})^n = 0.$$

The recursion that follows is:

$$\begin{aligned} a_n{}^{i+1} &= a_0{}^i \\ a_{n-1}{}^{i+1} &= a_1{}^i p_i + n a_0{}^i q_i \\ &\vdots \\ a_k{}^{i+1} &= \sum_{t=0}^{n-k} \binom{k+t}{t} a_{n-k-t}{}^i p_i{}^{n-k-t} q_i{}^t \qquad k = 0,1,\cdots,n. \end{aligned} \tag{7}$$

The resulting $i+1$th step polynomial equation is

$$P_n{}^{i+1} = a_n{}^{i+1} x_{i+1}{}^n + a_{n-1}{}^{i+1} x_{i+1}{}^{n-1} + \cdots + a_1{}^{i+1} x_{i+1} + a_0{}^{i+1} = 0.$$

The method to approximate the solution of (4) can be used now, if an algorithm for selecting p_i and q_i ($i = 1,2,\cdots$) is defined. This is the subject of Section V.

IV. INTERVALUES THAT ARE COVERED BY CONTINUED FRACTIONS

In this section we show how to approximate values of the solution by using different pairs of constants p_i and q_i.

Let p_i and q_i each assume two given values, e.g., $p_i,q_i \in \{a,b\}$. Continued fractions of the form (1) assume all values in the interval $[M,m]$, where

$$M = \max \lim_{k\to\infty} \frac{A_k}{B_k},$$

$$m = \min \lim_{k\to\infty} \frac{A_k}{B_k}.$$

A simple analysis shows that M can be found by solving a quadratic equation of the form

$$M = \frac{p_{mx}}{q_{\min} + p_{\min}/(q_{mx} + M)}$$

where $p_{mx},q_{mx} = \max(a,b)$, and $p_{\min},q_{\min} = \min(a,b)$.

Similarly, we have for m

$$m = \frac{p_{\min}}{q_{mx} + p_{mx}/(q_{\min} + m)}.$$

The resulting equations are

$$q_{\min}M^2 + (q_{\min}q_{mx} + p_{\min} - p_{mx})M - p_{mx}q_{mx} = 0$$

and

$$q_{mx}m^2 + (q_{mx}q_{\min} + p_{mx} - p_{\min})m - p_{\min}q_{\min} = 0.$$

Assuming that both p and q have the same range, e.g., $p_{\min} = q_{\min} = a$, $p_{mx} = q_{mx} = b$, $p,q \in \{a,b\}$, then the equation has the form

$$\begin{aligned} aM^2 + (ab + a - b)M - b^2 &= 0 \\ bm^2 + (ab + b - a)m - a^2 &= 0. \end{aligned} \tag{8}$$

In Table I we give values of a,b and the corresponding ranges $[m,M]$.

If a solution of the given polynomial equation is known to be in a certain interval, then the appropriate digit set a,b can be used to approximate this solution.

V. SELECTION RULES

In this section we develop a method for selecting p_i and q_i for the case $0 < a < b$, m and M positive. With some modifications the analysis can be developed for other cases. Assume that a pair of digits, $0 < a < b$, were selected according to the analysis of Section IV. The range of the solution, e.g., x_1, can be found now by solving (8). Since we are using a substitution of the form (5) our first condition will be

$$m \le x_i \le M \qquad i = 1,2,\cdots. \tag{9}$$

By imposing condition (9) we need now only one set of selection rules for p_i and q_i, $i = 1,2,\cdots$, in the range $[m,M]$.

We write below a version of (4). Let

$$x_1 = -a_0{}^1 \Big/ \sum_{k=1}^{n} a_k{}^1 x_1{}^{k-1}, \tag{10}$$

where it is assumed that $m \le x_1 \le M$.

TABLE I

a	b	m	M
$\frac{1}{2}$	1	$\frac{(2)^{1/2}-1}{2}$	$(2)^{1/2}$
1	2	$\frac{(17)^{1/2}-3}{4}$	$\frac{(17)^{1/2}-1}{2}$
-4	-6	1	2

We will find p_1 and q_1 such that

$$x_1 = \frac{p_1}{q_1 + x_2}, \tag{11}$$

where $m \leq x_2 \leq M$ and $p_1, q_1 \in \{a,b\}$. Clearly, we have four possibilities for selecting p_1 and q_1 and for each such pair we get different x_2. In order to make our selection we adopt the inverse approach. We assume that condition (9) exists for x_2, and find the range of x_1 for each pair of p_1 and q_1.

We start with the pair $p_1 = a$, $q_1 = b$. From (11) we have

$$C = \frac{a}{b+m} \geq x_1 \geq \frac{a}{b+M} = c. \tag{12}$$

Since a, b, m and M are known, C and c can be found and we have defined a range x_1 for which a selection of $p_1 = a$ and $q_1 = b$ will assure condition (9). Since x_1 is an unknown we substitute the results of (12) in (10) in order to find the allowable range for $p_1 = a$ and $q_1 = b$. We have

$$c \leq -a_0^1 / \sum_{k=1}^{n} a_k^1 x_1^{k-1} \leq C$$

and this result is possible for any x_1 in the range $[c,C]$, therefore we conclude that if the following two conditions are satisfied

$$\sum_{k=0}^{n} a_k^1 C^k \geq 0$$

$$\sum_{k=0}^{n} a_k^1 c^k \leq 0, \tag{13}$$

we select $p_1 = a$ and $q_1 = b$.

The analysis can be carried now for each of the remaining three pairs of values of p and q. Since x_2 has the same range as x_1 and satisfies a polynomial of the same degree, we can use the same procedure for x_2, etc.

The result is that the entire range is divided into four sections. For each section we can choose a pair of p_i and q_i, such that condition (9) will be satisfied for x_{i+1}.

Clearly, by using only the upper bound in (13), for each pair of p_i and q_i we can define a unique set of selection rules.

Our next objective is to show that there is an overlapping between two consecutive regions so that by using only the upper bound for each pair, the entire region is covered.

It can be verified that the four regions defined by (13) for each pair p_i and q_i are:

$$p = b, \quad q = a \tag{14a}$$

$$p = b, \quad q = b \tag{14b}$$

$$p = a, \quad q = a \tag{14c}$$

$$p = a, \quad q = b. \tag{14d}$$

In the theorem which follows, we give a necessary and sufficient condition, for the overlapping of the regions defined in (14), for two cases. The analysis for other values of a and b is similar and therefore is omitted.

Theorem 1: The regions defined by (13), for each of the pairs (14) overlap each other if and only if the following conditions are satisfied.

Condition 1: If $0 < a < b$, then $M \cdot m \leq \frac{1}{2}$.
Condition 2: If $b < a < 0$, then $M \cdot m \geq \frac{1}{2}$.

Proof: For $0 < a < b$ the regions defined in (14) are in decreasing order. Therefore we only have to show that the upper bound for the pair (14b)–(14d) is greater or equal to the lower bound of (14a)–(14c) respectively.

For the upper bound of (14b) and lower bound of (14a) we have,

$$\frac{b}{b+m} \geq \frac{b}{a+M}, \tag{15}$$

for the second pair we have,

$$\frac{a}{a+m} \geq \frac{b}{b+M}$$

and for the third

$$\frac{a}{b+m} \geq \frac{a}{a+M}. \tag{16}$$

The conditions that must be satisfied are:

$$M - m \geq b - a$$

$$M \cdot a \geq m \cdot b. \tag{17}$$

From the definition of a,b, the range $[m,M]$ and condition (9) we have

$$\frac{b}{a+m} = M \quad \text{and} \quad \frac{a}{b+M} = m.$$

Eliminating a and b we get

$$a = \frac{mM(1+m)}{1-mM}$$

and

$$b = \frac{mM(1+M)}{1-mM}.$$

The first condition in (17) is satisfied if and only if

$$M - m \geq b - a = \frac{mM(1 + M - m - 1)}{1 - mM}$$

$$= \frac{mM(M - m)}{1 - mM}$$

or

$$mM \leq \tfrac{1}{2}.$$

For the second condition we have,

$$\frac{M^2 m(1 + m)}{1 - mM} \geq \frac{m^2 M(1 + M)}{1 - mM}$$

or

$$M \geq m$$

which was assumed.

For the second case, $b < a < 0$, the inequalities in (15) and (16) reverse and therefore the result follows.

In the remaining part of this section we show how to approximate one solution of a given polynomial equation of order 5.

Let

$$P_5^1(x) = (x + 3)(x + 2)(x - 1)(x - 2)(x - 3)$$
$$= x^5 - x^4 - 13x^3 + 13x^2 + 36x - 36 = 0$$

be a given polynomial equation of order 5, and suppose that it is known that the interval

$$\left[\frac{(17)^{1/2} - 3}{4}, \frac{(17)^{1/2} - 1}{2}\right],$$

contains one solution.

Our first step is to select a pair of digits, a and b, according to the analysis of Section IV, such that every value in the given interval can be approximated by a continued fraction, $p,q \in \{a,b\}$. From Table I we have $a = 1$, $b = 2$.

The recursion relation between the coefficients of the ith and $i + 1$th polynomials of order 5 are:

$$a_5^{i+1} = a_0^i$$
$$a_4^{i+1} = a_1^i + 5a_0^i q_i$$
$$a_3^{i+1} = a_2^i p_i^2 + 4a_1^i p_i q_i + 10a_0^i q_i^2$$
$$a_2^{i+1} = a_3^i p_i^3 + 3a_2^i p_i^2 q_i + 6a_1^i p_i q_i^2 + 10a_0^i q_i^3$$
$$a_1^{i+1} = a_4^i p_i^4 + 2a_3^i p^3 q_i + 3a_2^i p_i^2 q_i^2 + 4a_1^i p_i q_i^3 + 5a_0^i q_i^4$$
$$a_0^{i+1} = a_5^i p_i^5 + a_4^i p_i^4 q_i + a_3^i p_i^3 q_i^2 + a_2^i p_i^2 q_i^3 + a_1^i p_i q_i^4 + a_0^i q_i^5.$$

The selection rules are

$$\text{for } P_5^i\left(\frac{5 - (17)^{1/2}}{2}\right) \geq 0, \quad p_i = 1, q_i = 2;$$

$$\text{for } P_5^i\left(\frac{(17)^{1/2} - 1}{4}\right) \geq 0, \quad p_i = 1, q_i = 1;$$

$$\text{for } P_5^i(5 - (17)^{1/2}) \geq 0, \quad p_i = 2, q_i = 2.$$

Otherwise $p_i = 2$, $q_i = 1$.

TABLE II

k	p_k	q_k	A_k	B_k	A_k/B_k	Error
1	2.0	1.0	0.0	0.1000D 01	0.0	0.10000D 01
2	2.0	1.0	0.2000D 01	0.1000D 01	0.2000000000000000D 01	-0.10000D 01
3	2.0	1.0	0.2000D 01	0.3000D 01	0.6666666666666666D 00	0.33333D 00
4	2.0	1.0	0.6000D 01	0.5000D 01	0.1200000000000000D 01	-0.20000D 00
5	2.0	1.0	0.1000D 02	0.1100D 02	0.9090909090909091D 00	0.90909D-01
6	2.0	1.0	0.2200D 02	0.2100D 02	0.1047619047619048D 01	-0.47619D-01
7	2.0	1.0	0.4200D 02	0.4300D 02	0.9767441860465116D 00	0.23256D-01
8	2.0	1.0	0.8600D 02	0.8500D 02	0.1011764705882353D 01	-0.11765D-01
9	2.0	1.0	0.1700D 03	0.1710D 03	0.9941520467836257D 00	0.58480D-02
10	2.0	1.0	0.3420D 03	0.3410D 03	0.1002932551319648D 01	-0.29326D-02
11	2.0	1.0	0.6820D 03	0.6830D 03	0.9985358711566618D 00	0.14641D-02
12	2.0	1.0	0.1366D 04	0.1365D 04	0.1000732600732601D 01	-0.73260D-03
13	2.0	1.0	0.2730D 04	0.2731D 04	0.9996338337605273D 00	0.36617D-03
14	2.0	1.0	0.5462D 04	0.5461D 04	0.1000183116645303D 01	-0.18312D-03
15	2.0	1.0	0.1092D 05	0.1092D 05	0.9999084500595075D 00	0.91550D-04
16	2.0	1.0	0.2185D 05	0.2185D 05	0.1000045777065690D 01	-0.45777D-04
17	2.0	1.0	0.4369D 05	0.4369D 05	0.9999771119910279D 00	0.22888D-04
18	2.0	1.0	0.8738D 05	0.8738D 05	0.1000011444135453D 01	-0.11444D-04
19	2.0	1.0	0.1748D 06	0.1748D 06	0.9999942779650155D 00	0.57220D-05
20	2.0	1.0	0.3495D 06	0.3495D 06	0.1000002861025678D 01	-0.28610D-05
21	2.0	1.0	0.6991D 06	0.6991D 06	0.9999985694892075D 00	0.14305D-05
22	2.0	1.0	0.1398D 07	0.1398D 07	0.1000000715255908D 01	-0.71526D-06
23	2.0	1.0	0.2796D 07	0.2796D 07	0.9999996423721740D 00	0.35763D-06
24	2.0	1.0	0.5592D 07	0.5592D 07	0.1000000178813945D 01	-0.17881D-06
25	2.0	1.0	0.1118D 08	0.1118D 08	0.9999999105930355D 00	0.89407D-07
26	2.0	1.0	0.2237D 08	0.2237D 08	0.1000000044703484D 01	-0.44703D-07
27	2.0	1.0	0.4474D 08	0.4474D 08	0.9999999776482583D 00	0.22352D-07
28	2.0	1.0	0.8948D 08	0.8948D 08	0.1000000011175871D 01	-0.11176D-07
29	2.0	1.0	0.1790D 09	0.1790D 09	0.9999999944120645D 00	0.55879D-08
30	2.0	1.0	0.3579D 09	0.3579D 09	0.1000000002793968D 01	-0.27940D-08
31	2.0	1.0	0.7158D 09	0.7158D 09	0.9999999986030161D 00	0.13970D-08
32	2.0	1.0	0.1432D 10	0.1432D 10	0.1000000000698492D 01	-0.69849D-09
33	2.0	1.0	0.2863D 10	0.2863D 10	0.9999999996507540D 00	0.34925D-09
34	2.0	1.0	0.5727D 10	0.5727D 10	0.1000000000174623D 01	-0.17462D-09
35	2.0	1.0	0.1145D 11	0.1145D 11	0.9999999999126885D 00	0.87311D-10
36	2.0	1.0	0.2291D 11	0.2291D 11	0.1000000000043656D 01	-0.43656D-10
37	2.0	1.0	0.4581D 11	0.4581D 11	0.9999999999781721D 00	0.21828D-10
38	2.0	1.0	0.9163D 11	0.9163D 11	0.1000000000010914D 01	-0.10914D-10
39	2.0	1.0	0.1833D 12	0.1833D 12	0.9999999999945430D 00	0.54570D-11
40	2.0	1.0	0.3665D 12	0.3665D 12	0.1000000000002728D 01	-0.27285D-11
41	2.0	1.0	0.7330D 12	0.7330D 12	0.9999999999986357D 00	0.13642D-11
42	2.0	1.0	0.1466D 13	0.1466D 13	0.1000000000000682D 01	-0.68212D-12
43	2.0	1.0	0.2932D 13	0.2932D 13	0.9999999999996589D 00	0.34106D-12
44	2.0	1.0	0.5864D 13	0.5864D 13	0.1000000000000170D 01	-0.17053D-12
45	2.0	1.0	0.1173D 14	0.1173D 14	0.9999999999999147D 00	0.85265D-13
46	2.0	1.0	0.2346D 14	0.2346D 14	0.1000000000000043D 01	-0.42633D-13
47	2.0	1.0	0.4691D 14	0.4691D 14	0.9999999999999787D 00	0.21316D-13
48	2.0	1.0	0.9382D 14	0.9382D 14	0.1000000000000010D 01	-0.10658D-13
49	2.0	1.0	0.1876D 15	0.1876D 15	0.9999999999999947D 00	0.53291D-14
50	2.0	1.0	0.3753D 15	0.3753D 15	0.1000000000000003D 01	-0.26645D-14

The numerical values for p_k, q_k, A_k, B_k, A_k/B_k and the error for the first fifty iterations are shown in Table II.

VI. CONVERGENCE CONDITIONS

In this section we develop necessary and sufficient conditions for the procedure to converge.

Theorem 2: Let

$$x_i = \frac{A}{B} = \frac{p_1}{q_1} + \frac{p_2}{q_2} + \cdots \frac{p_n}{q_n + x_{n+1}} \tag{18}$$

be a continued fraction representation of a solution of (4) which was found by substitution (5). $p_i,q_i \in \{a,b\}$ $x_i \in [m,M]$, $(i = 1,2,\cdots)$, where relations (8) exist between a, b, m and M.

Necessary and sufficient conditions for (18) to converge to a solution of the polynomial equation (4) are as follows.

Condition 1:

$$\frac{q}{p}(2m + a) > 0.$$

Condition 2:

$$2 + \frac{q}{p}(a + m - M) > 0.$$

Proof: Let A_n/B_n be the nth approximation to x_1. We will show that under conditions (1) and (2), for every $\epsilon > 0$, there exists N, such that for all $n > N$

$$\delta_n = \frac{A}{B} - \frac{A_n}{B_n} < \epsilon,$$

or equivalently that if $T_n = \delta_n/\delta_{n-2}$ $n = 3,4,\cdots$, then under the conditions of the theorem $|T_n| < 1$.

It was already shown in [1] that

$$T_n = \frac{1 - (q_n/p_n)x_n}{1 + (q_n/p_n)(B_{n-1}/B_{n-2})},$$

therefore the condition $|T_n| < 1$ implies

$$1 - \frac{q_n}{p_n}x_n < 1 + \frac{q_n}{p_n}\frac{B_{n-1}}{B_{n-2}}$$

and

$$-1 - \frac{q_n}{p_n}\frac{B_{n-1}}{B_{n-2}} < 1 - \frac{q_n}{p_n}x_n.$$

For the first expression we have

$$\frac{q_n}{p_n}\left(x_n + \frac{B_{n-1}}{B_{n-2}}\right) > 0,$$

and since this condition is true for every x_n it can be replaced by m, also min $(B_{n-1}/B_{n-2}) = a + m$ and the result is

$$\frac{q}{p}(a + 2m) > 0.$$

For the second expression we have

$$2 + \frac{q_n}{p_n}\left(\frac{B_{n-1}}{B_{n-2}} - x_n\right) > 0.$$

Here we substitute M for x_n and the minimum value of B_{n-1}/B_{n-2}, and the second condition follows:

$$2 + \frac{q}{p}(a + m - M) > 0.$$

VII. CONCLUSIONS

A method for the approximation of real roots of an n-order polynomial equation is described. The assumption is that intervals, each containing one solution, are known. Those intervals are later used according to the analysis of Section IV as m and M in order to define a pair of digits a and b, which will be used in a bilinear transformation such as (5) to approximate the solution.

The algorithm described in this paper consists of the following steps.

Step 1: Select a digit set, a and b, which covers one solution at a time.

Step 2: Use iteration (1) to approximate the solution.

Step 3: Iterate on (7) to get the coefficients of the $i + 1$th polynomial from the ith polynomial equation.

Step 4: Use selection rules, such as (13), which covers the entire allowable region, for the next p_i and q_i.

Step 5. Check for accuracy and if obtained divide A_k by B_k to receive one solution.

Step 6: Repeat Steps 1–5 for each interval which contains a zero of the given polynomial equation.

Notes

Note 1: If the condition of Theorem 1 is not satisfied for a certain interval, this interval can be divided into subintervals.

Note 2: By using powers of 2 for p_i and q_i, the time required to evaluate most of the expressions is reduced if a binary computer is used.

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Amnon Bracha-Barak, for a photograph and biography, see page 309 of the March 1974 issue of this TRANSACTIONS.