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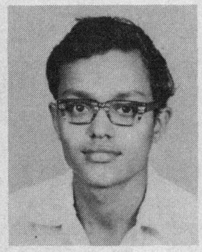

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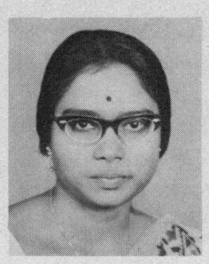

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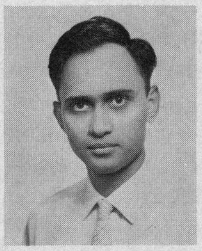

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Deterministic Division Algorithm in a Negative Base

P. V. SANKAR, S. CHAKRABARTI, AND E. V. KRISHNAMURTHY

***Abstract*—Described here is a deterministic division algorithm in a negative-base number system; here, the divisor is mapped into a suitable range by premultiplication, so that the choice of the quotient digit is deterministic.**

***Index Terms*—Deterministic division, negative base, range transformation.**

I. Introduction

Recently, deterministic division algorithms [1], [2] have been described for conventional and signed-digit number systems; these algorithms transform the divisor to a suitable range by premultiplication, so that the choice of the quotient digit is deterministic, without any need for a trial and error process. It is possible to develop a similar algorithm for division in a negative-base number system [3]. Let us denote the $(n+1)$ digit dividend as A, the $(d+1)$ digit divisor as B, and the $(m+1)$ digit quotient as Q in floating-point form (integral mantissa) in base $-\beta$. Thus

$$A = (-\beta)^{e_a} \cdot a = (-\beta)^{e_a} \sum_{j=0}^{n} a_j(-\beta)^j \tag{1a}$$

$$B = (-\beta)^{e_b} \cdot b = (-\beta)^{e_b} \sum_{j=0}^{d} b_j(-\beta)^j \tag{1b}$$

$$Q = (-\beta)^{e_q} \cdot q = (-\beta)^{e_q} \sum_{j=0}^{m} q_j(-\beta)^j \tag{1c}$$

where e_a, e_b, and e_q are the exponents and a, b, and q are the mantissas, respectively.

II. Notation and Definitions

The same notations and definitions as in [3] (for negative base) are used. However, for the sake of convenience, we define the following.

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Polarization: Polarization is an operation that reverses the sign of a number in the negative base. The polarized form of a is denoted by $\bar{a}$.

When an m-precision number is polarized, it could either contract to an $(m-1)$ precision number or expand to an $(m+1)$ precision number; in the latter case, the leading digit will be unity (this is defined as "polarization overflow").

III. Theory of Deterministic Division

For the purpose of our study, we denote by R_j the partial remainder at each step, with $R_j(j=n)=a$ the mantissa of the dividend.

Let us denote by a_j the digit in the $(-\beta)^j$th position of R_j. It is now interesting to study under what conditions the quotient digit q_j can be deterministically chosen as the leading digit of every R_j (or $\bar{R}_j$ when there is a polarization overflow) plus a suitable integer δ (positive or negative). In other words, the rules of choice are

$$q_j = a_j + \delta \text{ (positive)}$$

or

$$q_j = (-\beta + a_j) + \delta \text{ (negative)}.$$

The negative quotient arises when $\bar{R}_j$ has an overflow digit unity in the $(-\beta)^{j+1}$th position.

Assuming we carry out the recursion

$$\begin{aligned} R_{j-1} &= R_j - q_j b(-\beta)^{j-d} \\ &= R_j + q_j \bar{b}(-\beta)^{j-d} \end{aligned} \tag{2}$$

for $j=n, n-1, \cdots, n-m$, the choice of q_j as above will be the nonrestoring quotient if

$$|R_{j-1}| = |R_j - (a_j+\delta)b(-\beta)^{j-d}| < b(-\beta)^{j-d}. \tag{3}$$

This demands that b should be in the range

$$R_j/(a_j+\delta+1) < b(-\beta)^{j-d} < R_j/(a_j+\delta-1). \tag{4}$$

In order to find the range of b satisfying (4), we need to obtain the greatest lower bound and the least upper bound of the right- and left-hand side terms, respectively.

It is shown below that inequality (4) can be satisfied only for $\delta=0$ and $\delta=-1$.

Case 1: $\delta=0$.

$$(R_j/(a_j+1))_{\max} < b(-\beta)^{j-d} < (R_j/(a_j-1))_{\min}. \tag{5}$$

Note that

$$\begin{aligned} (R_j/(a_j+1))_{\max} &= \frac{(\beta-1)}{\beta}\,[(-\beta)^j + (-\beta)^{j-2} + \cdots \\ &\quad + (-\beta)^2 + 1] & (6) \\ &\mathrel{\overline{\overline{\cdot}}} (\beta/(\beta+1))(-\beta)^j & (7) \end{aligned}$$

while

$$\begin{aligned} (R_j/(a_j-1))_{\min} &= \frac{(\beta-1)(-\beta)^j + (\beta-1)}{(\beta-2)} \\ &\quad \cdot [(-\beta)^{j-1} + (-\beta)^{j-3} + \cdots + (-\beta)] & (8) \\ &\mathrel{\overline{\overline{\cdot}}} ((\beta^2-\beta-1)/(\beta+1)(\beta-2))(-\beta)^j & (9) \end{aligned}$$

(here $\mathrel{\overline{\overline{\cdot}}}$ denotes "approximately equal to").

Since $R_j=0$ implies $a_j=0$, we must omit this case while finding the bound in (8); in this case, we set $q_j=0$ (unlike the nonrestoring division scheme described in [3]).

Substituting (7) and (9) in (5), we get

$$(\beta/(\beta+1)) < b(-\beta)^{-d} < (\beta^2-\beta-1)/((\beta+1)(\beta-2)) \tag{10}$$

Case 2: $\delta=-1$.

In this case, (4) is rewritten as

$$(R_j/a_j)_{\max} < b(-\beta)^{j-d} < (R_j/(a_j-2))_{\min} \tag{11}$$

$$\begin{aligned} (R_j/a_j)_{\max} &= (-\beta)^j + (\beta-1) \\ &\quad \cdot [(-\beta)^{j-2} + \cdots + (-\beta)^2 + 1] & (12) \\ &\mathrel{\overline{\overline{\cdot}}} ((\beta+2)/(\beta+1))(-\beta)^j & (13) \end{aligned}$$

(when $R_j=0$, R_j/a_j is undetermined; we exclude this case to obtain (13) and take for this case, $q_j=0$) and

$$\begin{aligned} (R_j/(a_j-2))_{\min} &= \frac{(\beta-1)(-\beta)^j + (\beta-1)}{(\beta-3)} \\ &\quad \cdot [(-\beta)^{j-1} + \cdots + (-\beta)] & (14) \\ &\mathrel{\overline{\overline{\cdot}}} ((\beta^2-\beta-1)/(\beta+1)(\beta-3))(-\beta)^j. & (15) \end{aligned}$$

Substituting (13) and (15) in (11), we get

$$\begin{aligned} (\beta+2)/(\beta+1) &< b(-\beta)^{-d} \\ &< (\beta^2-\beta-1)/((\beta+1)(\beta-3)). \end{aligned} \tag{16}$$

Case 3: $\delta \geq 1$ and $\delta \leq -2$.

It is easy to see that, for these cases, inequality (4) cannot be satisfied for all R_j.

Allowed divisor patterns: Table I lists all leading three-digit patterns of b that satisfy (10) and (16). It is to be noted that the polarized form of the divisors listed in Table I also satisfy (10) and (16). Thus, initially it is necessary to check whether a given divisor or its polarized form satisfies Table I.

The case $\delta=-1$ has little practical utility, since range (16) is very much narrower than (10); hence, this case will not be considered.

Multiplier choice for obtaining the allowed patterns: The following rules are useful for constructing a table of two-digit multipliers $\alpha=\alpha_1(-\beta)+\alpha_0$ that would transform a given divisor b to satisfy (10), so that the leading three digits of the modified divisor (or its polarized form) assume one of the patterns specified in Table I for $\delta=0$. These rules can be arrived at by using the analysis available for positive-base numbers [4]–[6].

Rule 1:

TABLE I
ALLOWED DIVISOR PATTERNS

δ	b_d	b_{d-1}	b_{d-2}
	1	0	0
0	1	0	1
	1	1	$\gamma(\neq 0,1)$
-1	2	$(\beta-1)$	γ where $\gamma \leq \beta/2$ (β even) $\gamma \leq (\beta-1)/2$ (β odd)

a) If $b_d=1$, $b_{d-1}=0$, and $b_{d-2}=\gamma$ where $\gamma\neq 0$ or 1, then choose $\alpha_1=\beta-1$ and $\alpha_0=0$.

b) If $b_d=1$, $b_{d-1}=1$, and $b_{d-2}=0$ or 1, then choose $\alpha_1=2$ and $\alpha_0=\beta-1$.

c) If $b_d=1$ and $b_{d-1}\neq 1$ or 0, then polarize the divisor and use Rule 2.

d) If $b_d=2$ and $b_{d-1}=(\beta-1)$ and $b_{d-2}=0$, 1, or 2, then choose $\alpha_1=1$ and $\alpha_0=1$.

Rule 2: Otherwise, divide $(\beta^2-\beta)$ by the leading two digits (three digits if $b_d=1$) of b and obtain the two most significant digits of the quotient, viz, α_1 and α_0. Thus, if

$$((-\beta)^2-\beta)/(b_d(-\beta)+b_{d-1})=\alpha_1+\alpha_0/(-\beta)+\cdots \quad (17)$$

then choose $\alpha=\alpha_1(-\beta)+(\alpha_0-1)$.

IV. Steps of the Algorithm

Step 1: Prefix two leading zero digits to the dividend a to prevent initial overflow (see [3]); this dividend is denoted as R_n, the initial partial remainder; the number of digits in R_n is taken as $(n+1)$. Prefix one more leading zero to R_n to match the extra digit of the polarized divisor $\bar{b}$.

Step 2: Check whether the leading three digits of the divisor b(or $\bar{b}$) are among the patterns listed in Table I. If yes, go to Step 3; otherwise, choose a suitable multiplier α to transform the divisor into the required range.

Step 3: Set $q_n=1$ (to take care of initial overflow) and compute

$$R_{n-1} = R_n - b(-\beta)^{n-d}.$$

If there is no quotient overflow, then the $(n+1)$th, nth, and $(n-1)$th positions of R_{n-1} will contain 1, $(\beta-1)$, and 0, respectively. Then q_{n-1} equals $+\beta$, and this results in $q_n=0$ and $q_{n-1}=0$.

Otherwise, the $(n+1)$th and nth digit positions of R_{n-1} will be zero, and we take the $(n-1)$th digit of $R_{n-1}(=a_{n-1})$ as q_{n-1}.

Now onwards, if the $(j+1)$th $(j=n-2, n-1, \cdots)$ position of R_j is zero, choose the jth digit of R_j as q_j; otherwise, ($\overline{R}_j$ overflows) choose $(j+1)$th and jth digits of R_j as q_j; the quotient is then negative $(-\beta+a_j)$, and needs to be converted to the conventional form in negative base.

Then compute

$$R_{j-1} = R_j - q_j b(-\beta)^{j-d}$$

for each $j(=n-2, n-1, \cdots)$ and proceed to obtain q_j as above.

Step 4: Convert the negative quotient digits (if any) to positive form, using polarized addition.

Step 5: If we stop after obtaining q_{n-m}, then

$$e_q = e_a - e_b + n - m - d.$$

Example 1:

$$a = 14652;\ e_a = 0;\ b = 88;\ e_b = 0$$

$$\alpha = 27;\ \overline{b\cdot\alpha} = 1144;\ \overline{a\cdot\alpha} = 95236$$

$$n - m = 3;\ d = 3;\ q = 1909;\ e_q = 0.$$

$\vec{j}$		6	5	4	3	2	1	0	$q_j \downarrow$
R_6		0	0	9	5	2	3	6	1
$\overline{q_6\cdot b}$	1	9	0	7	6	0	0	0	
R_5			9	5	1	2	3	6	9
$\overline{q_5\cdot b}$		1	2	5	8	4	0	0	
R_4				0	9	6	3	6	0
$\overline{q_4\cdot b}$				0	0	0	0	0	
R_3					9	6	3	6	9
$\overline{q_3\cdot b}$				1	2	5	8	4	
R_2					0	0	0	0	

V. Concluding Remarks

It can be proved that, if we permit only those divisors whose leading three digits are of the form $11\gamma(\gamma\neq 0, 1, 2)$ in Table I, then a negative quotient digit cannot follow a positive quotient digit $(\beta-1)$. This eliminates the carry or borrow chain while converting the negative quotients. Hence, an economical left-to-right conversion of the quotient in parallel with the division process is possible. This will speed up the division process, since Step 4 of the algorithm need not be carried out separately.

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P. V. Sankar, for a photograph and biography, see this issue, page 125.

S. Chakrabarti, for a photograph and biography, see this issue, page 125.

E. V. Krishnamurthy, for a photograph and biography, see this issue, page 125.

A Video Display System for Image Processing by Computer

NORMAN H. KREITZER AND WILLIAM J. FITZGERALD

***Abstract*—A core-refreshed video display system that can display gray-scale images of 32 intensity levels on a standard monochrome video monitor will be described. The system can also display flicker-free black and white images of more than 800 000 picture elements. There are special features that allow overlaying black and white images on 16-level gray-scale images and manual cursor control via an *X-Y* tablet. Multiple reduced size images can be accommodated by features that allow independent manipulation of images in separate areas on the display screen. This permits simultaneous display of images before and after processing.**

***Index Terms*—Cathode-ray tube display, computer display, cursor, graphics, graphic tablet, image overlay, image processing, image refresh buffer, raster display, video display.**

I. INTRODUCTION

CATHODE-ray tube displays have been used extensively as computer output devices to communicate with alphanumeric or graphical symbols the results of information processing. For these symbols, only two screen intensities are required to represent black and white. Images with intermediate intensities, such as photographs, are now being processed by computer and additional display capability is required in order to present quickly the results of this information processing. A cathode-ray tube display system that can accommodate such gray-scale images will be described. The system can also display, without flicker, black and white images with greater detail than other computer output display systems known to the authors.

A signal and image processing experimental facility [1] that has been in existence at the T. J. Watson Research Center for a number of years, includes flying-spot scanners [2] and a tablet [3] for input of data to an IBM 1800 Computer. Output from the computer has been available via two displays and a Polaroid film recorder. One display, which creates images a dot at a time, is quite limited in the number of spots that it can display without flicker and is used primarily to show the results of scanning. This philosophy of display is developed further and described in [4]. The second display is a modified IBM 2250 [5] and can display only black and white. The scanners have the ability to enter gray-scale images into the 1800; and there are channel to channel connections from the 1800 to a 360 Model 67 and to a 360 Model 91, which can be used for image processing. There was a need for a display to provide quick access to the processed images.

Television technology was chosen for the display because: 1) the TV scan pattern is similar to that used by the optical scanners to scan images into the 1800; 2) a standardized scan pattern makes it unnecessary to store the positional data associated with the image; 3) flicker is not influenced by image complexity; and 4) considerable development had already been carried out in this area and some of the required equipment was readily available. TV technology also presented the opportunity to build a large black and white display that could be programmed to display arbitrary characters, symbols, and graphics. Since this display system was to be used for research, flexibility was a primary requirement. Because of the dual requirements for displaying both gray scale and black and white, the concept of a dual-line

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